8. Lecture 8: 3.1 Image and Kernel of a Linear Transformation

If $T\colon X \to Y$ is a transformation then the set X is called the **domain** of T. The set $\mathrm{Im}(T)$ of all images $T(\mathbf{x})$ when $\mathbf{x}$ varies over all points in the domain is called the **image** of T, or sometimes the **range**. Note that the image need not be all of the **target space** Y.

T is said to be **onto** if each $\mathbf{y} \in Y$ is the image $T(\mathbf{x})$ of at least one $\mathbf{x} \in X$.
T is said to be **one-to-one** if each $\mathbf{y} \in Y$ is the image of at most one $\mathbf{x} \in X$.
T is called **invertible** if its one-to-one and onto.

Ex 7 Define $T\colon \mathbf{R}^2 \to \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Is T onto? What is the image?

Sol The image of T is all combinations of the column vectors of A

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} x_2$$

for any x_1 and x_2. The image is the plane 'spanned' by the two column vectors.

Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ and scalars $\lambda_1, \dots, \lambda_k$, the vector

$$\mathbf{w} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, with weights $\lambda_1, \dots \lambda_k$.

The set of all linear combinations of a $\mathbf{v}_1, ..., \mathbf{v}_n$ is called the **span** of $\mathbf{v}_1, ..., \mathbf{v}_n$ and is denoted by $\mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$. The set $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ **span** (is a **spanning set** for) V if every vector in V can be written as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$.

Th The image of a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is the span of the column vectors of A.
Pf

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n$$

The image of a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is also called the column space of A, $\mathrm{Col}(A)$.

A **subspace** W of $\mathbf{R}^n$ is a subset which is closed under addition and scalar multiplication: (a) $\mathbf{0} \in W$, (b) $\mathbf{u} \in W$ and $\mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$, (c) $\mathbf{w} \in W$ and k is a scalar then $k\mathbf{w} \in W$.

Ex A plane $ax_1 + bx_2 + cx_3 = 0$ going through the origin in space is a subspace of $\mathbf{R}^3$.

Th The image of a linear transformation $T(\mathbf{x}) = A\mathbf{x}$, from $\mathbf{R}^n \to \mathbf{R}^m$ is a subspace of $\mathbf{R}^m$.
Pf For a proof see the proof of Theorem 3.1.4 in the textbook.
Alternatively it follows from the previous theorem and the following theorem:

Th If $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{R}^m$ then $\mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$ is a subspace of $\mathbf{R}^m$.
Pf (b) follows from that sums of linear combinations are linear combination. In fact let $W = \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$. Then if $\mathbf{u} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n \in W$ and $\mathbf{w} = d_1 \mathbf{v}_1 + \cdots + d_n \mathbf{v}_n \in W$ it follows that $\mathbf{u} + \mathbf{w} = (c_1 + d_1)\mathbf{v}_1 + \cdots + (c_n + d_n)\mathbf{v}_n \in W$ since it is also a linear combination.

The kernel

The **kernel**, Ker(T), of a linear transformation $T:\mathbf{R}^n\to\mathbf{R}^m$ is the set of all $\mathbf{x}$ in the domain such that $T(\mathbf{x})=0$. It is a proper subset of the domain $\mathbf{R}^n$ unless T is the zero map.

Ex Let $T:\mathbf{R}^3\to\mathbf{R}^2$ by $T(\mathbf{x})=A\mathbf{x}$, where $A=\begin{bmatrix}1&2&0\\0&1&1\end{bmatrix}$. Is T one-to-one? What is the kernel?

Sol $A\mathbf{x}=\mathbf{0}$ has nontrivial solutions since there are more variables than equations. Hence there are infinitely many points such that $T(\mathbf{x})=0$ so T is not one-to-one. Explicitly

$$\begin{bmatrix}1&2&0&0\\0&1&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2&0\\0&1&1&0\end{bmatrix},\quad\Leftrightarrow\quad\begin{aligned}x_1&=2x_3\\x_2&=-x_3\\x_3&=\text{free}\end{aligned}$$

The kernel is hence the subspace spanned by the line $\mathbf{x}=\begin{bmatrix}2\\-1\\1\end{bmatrix}t$, for any parameter t.

The kernel of a linear transformation $T(\mathbf{x})=A\mathbf{x}$ is also called the null space of A, Nul(A)

Th The kernel of a linear transformation $T(\mathbf{x})=A\mathbf{x}$, from $\mathbf{R}^n\to\mathbf{R}^m$ is a subspace of $\mathbf{R}^n$.
Pf We must verify the three properties (a), (b), (c) in the definition of subspace.
(a) $\mathbf{0}\in\text{Nul }A$ since $A\mathbf{0}=\mathbf{0}$.
(b) If $\mathbf{u},\mathbf{v}\in\text{Nul }A$, show that $\mathbf{u}+\mathbf{v}\in\text{Nul }A$. $A(\mathbf{u}+\mathbf{v})=A\mathbf{u}+A\mathbf{v}=\mathbf{0}+\mathbf{0}=\mathbf{0}$.
(c) If $\mathbf{u}\in\text{Nul }A$, show that $\lambda\mathbf{u}\in\text{Nul }A$. $A(\lambda\mathbf{u})=\lambda A\mathbf{u}=\lambda\mathbf{0}=\mathbf{0}$.

Ex 1 Find an **explicit description** of Nul A where $A=\begin{bmatrix}3&6&6&3&9\\6&12&13&0&3\end{bmatrix}$.

Sol Row reduction to solve $A\mathbf{x}=0$; $\begin{bmatrix}3&6&6&3&9&0\\6&12&13&0&3&0\end{bmatrix}\sim\ (1)/3\begin{bmatrix}1&2&2&1&3&0\\6&12&13&0&3&0\end{bmatrix}\sim$

$$(2)-6(1)\begin{bmatrix}1&2&2&1&3&0\\0&0&1&-6&-15&0\end{bmatrix}\sim\ (1)-2(2)\begin{bmatrix}1&2&0&13&33&0\\0&0&1&-6&-15&0\end{bmatrix}$$

Hence $A\mathbf{x}=\mathbf{0}\Leftrightarrow\begin{cases}x_1+2x_2\ +\ 13x_4+33x_5=0\\ \qquad\qquad x_3-6x_4-15x_5=0\end{cases}$. x_2,x_4,x_5 are free so the sol. is

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-2x_2-13x_4-33x_5\\x_2\\6x_4+15x_5\\x_4\\x_5\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-13\\0\\6\\1\\0\end{bmatrix}+x_5\begin{bmatrix}-33\\0\\15\\0\\1\end{bmatrix}$$

Hence Nul $A=\text{Span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$, is the span of the three vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ above.

We always have that $\mathbf{0} \in \text{Ker}(A)$. When is $\text{Ker}(A) = \{0\}$?

Th (a) If A is $m \times n$ then $\text{Ker}(A) = \{0\}$ if and only if $\text{rank}(A) = n$.
(b) If A is $m \times n$ and $\text{Ker}(A) = \{0\}$ then $m \leq n$.
(c) If A is $n \times n$ then $\text{Ker}(A) = \{0\}$ if and only if A is invertible.

Th For an $n \times n$ matrix A the following statements are equivalent:
(i) A is invertible
(ii) $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for all $\mathbf{b}$.
(iii) $\text{Rref}(A) = I$.
(iv) $\text{rank}(A) = n$.
(v) $\text{Im}(A) = \mathbf{R}^n$.
(vi) $\text{Ker}(A) = \mathbf{0}$

Summary and Questions

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Th The kernel of a linear transformation $T(\mathbf{x}) = A\mathbf{x}$, from $\mathbf{R}^n \to \mathbf{R}^m$ is a subspace of $\mathbf{R}^n$.

Question Which of the following are subspaces: (a) the plane $x_1 + 2x_2 - 4x_3 = 1$, (b) The span of the vectors $(1, 2, 4)$ and $(2, 4, 8)$? (c) The Kernel of the matrix corresponding to rotation by 90 degrees counterclockwise? (d) The circle $x_1^2 + x_2^2 = 1$. (e) The ball $x^2 + y^2 + z^2 \leq 1$.

Question Which of the following transformations have a nontrivial kernel (i.e. containing more than just $\mathbf{0}$)? (a) Rotation by $\pi/2$ counterclockwise, (b) Projection of the plane onto the x axis. (c) Reflection in the x axis.