9. Lecture 9: 3.2 Linear Independence and Bases

Let us start by recalling some definitions and facts that we proved:

A **subspace** W of $\mathbf{R}^m$ is a subset which is closed under addition and scalar multiplication: (a) $\mathbf{0} \in W$, (b) $\mathbf{u} \in W$ and $\mathbf{w} \in W$ then $\mathbf{u}+\mathbf{w} \in W$, (c) $\mathbf{w} \in W$ and $k \in \mathbf{R}$ then $k\,\mathbf{w} \in W$.

The examples of subspaces we know are either images or kernels of linear transformations:
Th If $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation, from $\mathbf{R}^n \to \mathbf{R}^m$, then
(a) The Image of T, also called the Column space of A, is a subspace of $\mathbf{R}^m$.
(b) The Kernel of T, also called the Null space of A is a subspace of $\mathbf{R}^n$.

The **span** of $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{R}^m$, denoted $\mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$, is the set of a all **linear combinations**

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k, \qquad \text{with} \quad \lambda_1, \ldots, \lambda_n \in \mathbf{R}.$$

Th If $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{R}^m$ then $W = \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$ is a subspace of $\mathbf{R}^m$.
Pf If $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \in W$ and $\mathbf{w} = d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n \in W$ it follows that $\mathbf{u}+\mathbf{w} = (c_1+d_1)\mathbf{v}_1 + \cdots + (c_n+d_n)\mathbf{v}_n \in W$ since it is also a linear combination.

As we shall see every subspace W of $\mathbf{R}^m$ is the span $\mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$ of some vectors in $\mathbf{R}^m$.

Question Could it be that a subset of the vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ span $\mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$?

Ex. Give a minimal set of vectors spanning the column space of the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix}$

Sol. The 3 column vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$ span the column space by definition. It is also clear from inspection that none of them can be written as a multiple of another alone. Hence only the possibility that one of them can be written as a linear combination of the other two remains:

$$\mathbf{v}_1 = c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3, \quad \text{or} \quad \mathbf{v}_2 = c_{21}\mathbf{v}_1 + c_{23}\mathbf{v}_3, \quad \text{or} \quad \mathbf{v}_3 = c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2. \tag{9.1}$$

The statement, that one of these 3 equalities hold, is equivalent to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}, \quad \text{for some } c_1, c_2, c_3 \text{ not all } 0. \tag{9.2}$$

In fact if one of the equations in (9.1), say the first, holds then $\mathbf{v}_1 - c_{12}\mathbf{v}_2 - c_{13}\mathbf{v}_3 = \mathbf{0}$ so (9.2) holds with $c_1 = 1$, $c_2 = -c_{12}$, $c_3 = -c_{13}$. On the other hand if (9.2) holds with one of c_1, c_2, c_3 different from 0, say $c_1 \neq 0$, then dividing by c_1 we get $\mathbf{v}_1 = -(c_2\mathbf{v}_2 + c_2\mathbf{v}_3)/c_1$ so (9.1) holds. However, (9.2) is equivalent to that the system $A\mathbf{c} = \mathbf{0}$ has a nontrivial solution $\mathbf{c} \neq 0$:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is free there are nontrivial solutions $x_1 = -33x_3$, $x_2 = 18x_3$, x_3 is free. If we e.g. let $x_3 = 1$ then $x_1 = -33$ and $x_2 = 18$ so we have the linear dependence relation

$$-33\mathbf{v}_1 + 18\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

Hence $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathrm{Span}(\mathbf{v}_2, \mathbf{v}_3) = \mathrm{Span}\ (\mathbf{v}_3, \mathbf{v}_1)$.

We say that a vector $\mathbf{v}_i$, in an ordered list $\mathbf{v}_1, \dots, \mathbf{v}_n$, is **redundant** if it is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$, i.e. for some constants $c_1, \dots, c_{i-1}$ we have

$$\mathbf{v}_i = c_1\mathbf{v}_1 + \cdots + c_{i-1}\mathbf{v}_{i-1}. \tag{9.3}$$

We say that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are **linearly independent** if none of them is redundant.
If at least one is redundant they are said to be **linearly dependent**.
The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to satisfy a **nontrivial linear relation** if

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_p\mathbf{v}_p = \mathbf{0}, \qquad \text{for some } \lambda_1, \dots, \lambda_p, \text{ not all } 0. \tag{9.4}$$

Th $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ are linearly dependent if and only if they satisfy a nontrivial linear relation.
Pf It is easy to see that (9.4) is equivalent to (9.3) for some i. First if (9.4) hold take the largest k such that $\lambda_k \neq 0$. Then $\mathbf{v}_k = -(\lambda_1\mathbf{v}_1 + \cdots + \lambda_{k-1}\mathbf{v}_{k-1})/\lambda_k$ so (9.3) hold. On the other hand if (9.3) hold then $\mathbf{v}_i - c_1\mathbf{v}_1 - \cdots - c_{i-1}\mathbf{v}_{i-1} = 0$ so (9.4) hold.

Usually (9.4) is taken as definition of linear dependence and linear independence is that

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0} \tag{9.5}$$

has only the trivial solution $x_1 = \cdots = x_p = 0$. (9.5) can be written in matrix form

$$A\mathbf{x} = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_p \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p,$$

and the linear independence of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is that this equation has only the trivial solution $\mathbf{x} \neq \mathbf{0}$, whereas linear dependence is the statement that it has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$.

Let V be a subspace of $\mathbf{R}^m$. We say that the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ form a **basis** for V if they span V and are linearly independent.

Th $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ form a basis for V if (and only if) every vector $\mathbf{v} \in V$ can be expressed **uniquely** as a linear combination $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m$ (i.e. for unique $c_1, \dots, c_m$).
Pf Only the uniqueness remains to be shown. If we had two representations $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = d_1\mathbf{v}_1 + \cdots + d_m\mathbf{v}_m$, then their difference would satisfy $(c_1 - d_1)\mathbf{v}_1 + \cdots + (c_m - d_m)\mathbf{v}_m = \mathbf{0}$. However, since the vectors are linearly independent this would imply that $c_1 = d_1, \dots c_m = d_m$ so it was unique after all.

Th Let W be a subspace of $\mathbf{R}^m$. Then W has a basis, i.e. $W = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$ of some linearly independent vectors.
Pf If $W \neq \{\mathbf{0}\}$ there is a $\mathbf{v}_1 \in W \setminus \{0\}$. Let $W_1 = \mathrm{Span}(\mathbf{v}_1)$. If $W \setminus W_1 \neq \emptyset$ then we can take $\mathbf{v}_2 \in W \setminus W_1$ and define $W_2 = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$, and so on if $W \setminus W_{k-1} \neq \emptyset$ we take $\mathbf{v}_k \in W \setminus W_{k-1}$ and form $W_k = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. By construction $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent. We claim that this process has to stop at some point with $k \leq m$. In fact suppose not and $k > m$. Then $\lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k = \mathbf{0}$, would have a nontrivial solution since it has more unknowns than equations. This would contradict the linear independence proving that $k \leq m$.

The theorem shows that any proper subspace of $\mathbf{R}^3$ is either spanned by one vector, and is a line through the origin, or spanned by two vectors, and is a plane through the origin.

Summary and Questions

A **subspace** W of $\mathbf{R}^m$ is a subset which is closed under addition and scalar multiplication: (a) $\mathbf{0} \in W$, (b) $\mathbf{u} \in W$ and $\mathbf{w} \in W$ then $\mathbf{u}+\mathbf{w} \in W$, (c) $\mathbf{w} \in W$ and $k \in \mathbf{R}$ then $k\,\mathbf{w} \in W$.

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$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k, \qquad \text{with} \quad \lambda_1, \ldots, \lambda_n \in \mathbf{R}.$$

Th If $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{R}^m$ then $W = \text{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$ is a subspace of $\mathbf{R}^m$.
Th Let W be a subspace of $\mathbf{R}^m$. Then $W = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ of some vectors.

Th If $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation, from $\mathbf{R}^n \to \mathbf{R}^m$, then
(a) The Image of T, also called the Column space of A, is a subspace of $\mathbf{R}^m$.
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Th $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ are linearly dependent if and only if they satisfy a nontrivial linear relation.

Usually (9.6) is taken as definition of linear dependence and linear independence is that

$$x_1 \mathbf{v}_1 + \cdots + x_p \mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = \cdots = x_p = 0$. This can be written in matrix form

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