Solutions to Math 201 Midterm I Fall 07, Zucker

1. a) The $\mathcal{B}$-matrix for T is the matrix B of the linear transformation T expressed in the $\mathcal{B}$ coordinates $[T(\mathbf{x})]_{\mathcal{B}} = B\,[\mathbf{x}]_{\mathcal{B}}$, where $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ are the coordinates of $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$ expressed in the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. To find $B = \begin{bmatrix} | & | & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & [T(\mathbf{b}_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}$ we calculate $T(\mathbf{b}_i)$ and express them in the $\mathcal{B}$ coordinates:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1&1&0\\0&1&1\\0&0&2\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \text{i.e.} \qquad \left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1&1&0\\0&1&1\\0&0&2\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}, \qquad \text{i.e.} \qquad \left[T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1&1&0\\0&1&1\\0&0&2\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\1\end{bmatrix}, \qquad \text{i.e.} \qquad \left[T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}1\\0\\2\end{bmatrix}$$

Hence $B = \begin{bmatrix}1&0&1\\0&1&0\\0&1&2\end{bmatrix}$.

b) By linearity $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(2\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}\right) = 2T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = 2\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}$

2. a) We apply Gauss-Jordan elimination to the augmented matrix for the system:

$$\begin{bmatrix}2&1&1&-2&3\\1&1&0&1&2\\1&2&-1&4&5\end{bmatrix} \Leftrightarrow \begin{bmatrix}1&1&0&1&2\\2&1&1&-2&3\\1&2&-1&5&4\end{bmatrix} \Leftrightarrow \begin{bmatrix}1&1&0&1&2\\0&-1&1&-4&-1\\0&1&-1&4&2\end{bmatrix} \Leftrightarrow \begin{bmatrix}1&1&0&1&2\\0&1&-1&4&1\\0&0&0&0&1\end{bmatrix} \Leftrightarrow \begin{bmatrix}1&0&1&-3&1\\0&1&-1&4&1\\0&0&0&0&1\end{bmatrix}.$$

Since the last equation in the inhomogeneous case would say $0 = 1$ it is inconsistent. The reduction however gives the $rref(A)$ as the 4 first columns of this matrix.

b) Moreover the same calculation in the inhomogeneous case $A\mathbf{x} = \mathbf{0}$ gives that $\mathbf{x}$ is in the kernel of A if $x_1 + x_3 - 3x_4 = 0$ and $x_2 - x_3 + 4x_4 = 0$ where x_3 and x_4 are free so we obtain $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-x_3+3x_4\\x_3+4x_4\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}x_3 + \begin{bmatrix}3\\-4\\0\\1\end{bmatrix}x_4$ so the vectors $\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\-4\\0\\1\end{bmatrix}$ form a basis for the kernel.

c) The dimension of the image of A can be obtain from the fundamental theorem of algebra:

$$\dim\big(\operatorname{im} A\big) + \dim\big(\ker A\big) = \dim\big(\text{domain } A\big)$$

once we know that the dimension of the kernel of A is 2 and the dimension of the domain of A i.e. $\mathbb{R}^4$ is 4 so dimension of the image of A is 2. Alternatively, the columns of A corresponding to columns of $rref(A)$ with a leading one form a basis for the image of A. Hence the first two columns of A form a basis for the image of A so it has dimension 2.

3 a) That $\mathbf{u}, \mathbf{v} \in Y$ implies that $\mathbf{u}+\mathbf{v} \in Y$ and $\lambda\mathbf{u} \in Y$.

b) $\dim Y$ is the number of elements in a basis for Y.

c) If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in Y$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in Y$ then $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \\ u_4+v_4 \end{bmatrix}$ satisfy
$2x_1 + x_2 + x_3 - 2x_4 = 2(u_1+v_1) + u_2+v_2+u_3+v_3 - 2(u_4+v_4)$
$= 2u_1 + u_2 + u_3 - 2u_4 + 2v_1 + v_2 + v_3 - 2v_4 = 0 + 0 = 0$ so $\mathbf{x} = \mathbf{u}+\mathbf{v} \in Y$.
One similarly proves that $\lambda\mathbf{u} \in Y$.

d) The *rref* of the system is $x_1 + x_2/2 + x_3/2 - x_4 = 0$. Hence x_2, x_3, x_4 are free variables and the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$ so $\begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for Y. We claim that these 3 vectors, $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ together with any vector that is not in Y, e.g. $\mathbf{b}_4 = \mathbf{e}_4$ of the standard basis, form a basis for $\mathbb{R}^4$. To prove this it is sufficient to show that they are linearly independent since 4 vectors in $\mathbb{R}^4$ form a basis if they are linearly independent. In fact if $\lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2 + \lambda_3\mathbf{b}_3 + \lambda_4\mathbf{b}_4 = \mathbf{0}$ then if $\lambda_4 \neq 0$ it follows that $\mathbf{b}_4 = -\big(\lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2 + \lambda_3\mathbf{b}_3\big)/\lambda_4 \in Y$, which is a contradiction so we must have that $\lambda_4 = 0$. However then we must have $\lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2 + \lambda_3\mathbf{b}_3 = \mathbf{0}$, but since $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent it follows that also $\lambda_1 = \lambda_2 = \lambda_3 = 0$ so $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ are linearly independent.

4. $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$ so $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

5. We need to determine the dimension of the image of $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 1 & 0 & -1 & 1 \end{bmatrix}$.

Row reduction gives $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 1 & 0 & -1 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & -1 & -1 & -1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
The columns of A corresponding to columns of *rref*(A) with a leading one form a basis for the image of A. Hence the first two columns of A form a basis for the image of A so it has dimension 2.

6. a) If $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and they all have rank 1.
The matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ both have rank 1 but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has rank 0.

b) We have that $\mathrm{Im}(AB) \subset \mathrm{Im}(A)$. In fact if $\mathbf{y} \in \mathrm{Im}(AB)$ then $\mathbf{y} = AB\mathbf{x}$ for some $\mathbf{x}$ so then $\mathbf{y} = A\mathbf{z}$, where $\mathbf{z} = B\mathbf{x}$ and so $\mathbf{y} \in \mathrm{Im}(A)$. Since the rank is the dimension of the image, we must therefore have that $\mathrm{rank}(AB) \leq \mathrm{rank}(A) \leq 1$.

c) We have that $\mathrm{Ker}(B) \subset \mathrm{Ker}(AB)$. In fact if $\mathbf{x} \in \mathrm{Ker}(B)$ then $B\mathbf{x} = \mathbf{0}$ so $AB\mathbf{x} = \mathbf{0}$ so then $\mathbf{x} \in \mathrm{Ker}(AB)$. With the nullity denoting the dimension of the kernel it follows that $\mathrm{nullity}(B) \leq \mathrm{nullity}(AB)$. Using the fundamental theorem of linear algebra:

$$\dim\big(\mathrm{im}\, C\big) + \dim\big(\ker C\big) = \dim\big(\mathrm{domain}\, C\big)$$

applied to B gives that $\mathrm{rank}(B) + \mathrm{nullity}(B) = 2$ and since by assumption $\mathrm{rank}(B) = 1$ it follows that $\mathrm{nullity}(B) = 1$ and therefore $\mathrm{nullity}(AB) \geq 1$ and applying the fundamental theorem of linear algebra to AB gives that $\mathrm{rank}(AB) + \mathrm{nullity}(AB) = 2$ so $\mathrm{rank}(AB) \leq 1$.

7. *False*: T could e.g. just be the zero transformation.