Johns Hopkins University
Math 201, Spring 2007
Name: Solutions
Section:

Midterm Exam # 1

Time: 50 minutes

No books, notes, calculators. Please explain carefully all steps leading to your solutions, or risk losing credit.

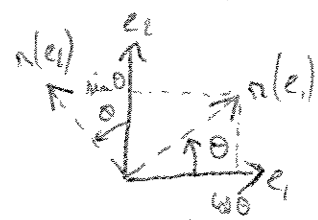
Problem 1: (4 points=2+2)

1. Let $r : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation of angle θ around the origin. Write the matrix corresponding to r. Justify it by a drawing.

2. Let $R : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the rotation of angle θ around the line spanned by $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Write the matrix corresponding to R. Justify it by a drawing.

1) We find the image of the standard vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$:

$r(e_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\quad r(e_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ as can be seen on the drawing



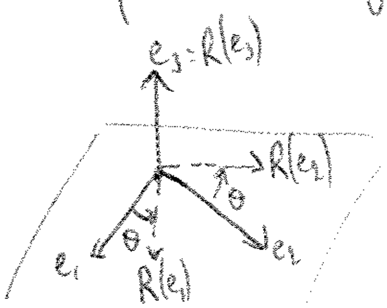
The standard matrix of r is therefore:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

2) We find the image of the standard vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$:

$R(e_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$ $\quad R(e_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$ and $R(e_3) = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(R leaves e_3 fixed and is a rotation through θ in the plane spanned by e_1 and e_2)



The standard matrix of R is therefore:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2: (8 points=4+2+1+1)
Consider the matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

1. Is A invertible? If yes, compute its inverse and check your answer by evaluating the matrix product AA^{-1}.

2. Solve the linear system $AX = Y$ for $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

3. What is rref(A)?

4. What is the rank of A?

1) We use Gauss-Jordan elimination to put A in reduced row echelon form (and do the same operations on I_3 to find A^{-1} if it exists):

$$\begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -2\times L_1 \\ -2\times L_1 \end{array} \quad \text{then exchange } L_2 \text{ and } L_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \\ 0 & -5 & -1 & -2 & 1 & 0 \end{array}\right) \begin{array}{l} +L_2 \\ \times(-1/3) \\ -5/3 \times L_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2/3 & 0 & -1/3 \\ 0 & 0 & -1 & 4/3 & 1 & -5/3 \end{array}\right) \begin{array}{l} +L_3 \\ \\ \times(-1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1 & -2/3 \\ 0 & 1 & 0 & 2/3 & 0 & -1/3 \\ 0 & 0 & 1 & -4/3 & -1 & 5/3 \end{array}\right)$$

the left-hand side is rref$(A) = I_3$. This shows that A is invertible, with:

$$A^{-1} = \begin{pmatrix} 1/3 & 1 & -2/3 \\ 2/3 & 0 & -1/3 \\ -4/3 & -1 & 5/3 \end{pmatrix}$$

. We check this result by evaluating:

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1/3 & 1 & -2/3 \\ 2/3 & 0 & -1/3 \\ -4/3 & -1 & 5/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) Since we know A^{-1}, we can simply solve the system $AX=Y$ by multiplying each side by A^{-1} to obtain: $X = A^{-1} \cdot Y$.

Therefore:

$$X = \begin{pmatrix} 1/3 & 1 & -2/3 \\ 2/3 & 0 & -1/3 \\ -4/3 & -1 & 5/3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/3 + 1 - 2 \\ 4/3 - 1 \\ -8/3 - 1 + 5 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ 4/3 \end{pmatrix}$$

3) We have seen that $\mathrm{rref}(A) = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (This is equivalent to the fact that A is invertible)

4) The rank of A is 3 (as all invertible 3×3 matrices).

Problem 3: (8 points=1+4+3)
Consider the matrix:

$$B = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$$

1. What are the domain and codomain of the linear transformation f defined by B?
2. Find a basis of the image $\mathrm{Im}(B)$. What is the rank of B?
3. Find a basis of the kernel $\mathrm{Ker}(B)$. What is its dimension? How could you predict this value?

1) B is a 3×4 matrix, so $f : \underset{\text{domain}}{\mathbb{R}^4} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^3}$

2) Consider the column vectors v_1, v_2, v_3, v_4 of B.

$*$ $v_1 \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so v_1 is not redundant.

$*$ v_2 is not a scalar multiple of v_1, so v_2 is not redundant.

$*$ $v_3 = v_1 + v_2$ is redundant

$*$ $v_4 = -\frac{2}{7} \cdot v_1 + \frac{4}{7} v_2$ is also redundant

(this is seen by solving the system: $v_4 = x_1 \cdot v_1 + x_2 v_2$).

Therefore a basis of $\mathrm{Im}(B)$ is (v_1, v_2). The rank of B is 2 (it is the dimension of $\mathrm{Im}(B)$

3) We have found the two relations among the column vectors of B:

$$\begin{cases} v_3 = v_1 + v_2 & \longrightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \text{ is in } \mathrm{Ker}(B) \\ v_4 = -\frac{2}{7} v_1 + \frac{4}{7} v_2 & \longrightarrow \begin{pmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{pmatrix} \text{ is in } \mathrm{Ker}(B) \end{cases}$$

We know that a basis of $\mathrm{Ker}(B)$ is therefore $\left(\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{pmatrix} \right)$.

The dimension of $\mathrm{Ker}(B)$ is 2.

We could predict this value from the rank-nullity theorem, which says that: $\dim(\mathrm{Ker}\, B) + \dim(\mathrm{Im}(B)) = \dim(\mathrm{domain}(B)) = 4$.