THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences

110.201 - Linear Algebra
Midterm Exam - Spring Session 2010

Instructions: This exam has 13 pages. No calculators, books or notes allowed.

• **You must answer the first 3 questions, then answer one of question 4 or 5. Do not answer both. No extra points will be rewarded.**

Question 6 is bonus.

• **Place an "X" through the question you are not going to answer.**

• **You must use a pen.**

You have: 50 MINUTES.

Be sure to show all work for all problems. No credit will be given for answers without work shown. If you do not have enough room in the space provided you may use additional paper: **ask the Instructor to get additional paper**. If you use extra paper, be sure to clearly label each problem and attach the extra paper to the exam.

Academic Honesty Certification

I agree to complete this exam without unauthorized assistance from any person, materials or device.

Signature: ______________________________ Date: ____________________

Name of the student **AND session nb. (or TA's name)**: ______________________________

Problem	Score
1	
2	
3	
4 or 5	
6 (Bonus)	
Total	

1a. [15 points] Given the following system of equations:

$$\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$$

find all solutions using Gauss-Jordan elimination procedure. Is this an example of a consistent system? Why?

1b. [10 points] Can the following system of equations:

$$\begin{aligned} x - y + z &= 1 \\ x + y - 2z &= 2 \\ 2x - z &= 3 \end{aligned}$$

have a single solution? Justify your answer.

2. [25 points] Given the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

Can the system

$$A \cdot \overrightarrow{x} = \overrightarrow{b}, \qquad \overrightarrow{x} \in \mathbb{R}^5$$

have infinitely many solutions for some vector $\overrightarrow{b} \in \mathbb{R}^4$ and be inconsistent for some other vector $\overrightarrow{b_1} \in \mathbb{R}^4$? If yes, give two such vectors. Explain your answer.

Hint. Study the reduced row-echelon form of the matrix $(A \mid \overrightarrow{b})$.

3. [25 points] Given the following transformations:

(a) $f_1 : \mathbb{R}^2 \to \mathbb{R}^2 \quad f_1(x_1, x_2) = (x_1 + 2x_2, e^{x_1+x_2})$

(b) $f_2 : \mathbb{R}^3 \to \mathbb{R}^5 \quad f_2(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3, 3x_1 + x_2, x_1, 5x_3)$

(c) $f_3 : \mathbb{R}^3 \to \mathbb{R}^3 \quad f_3(x_1, x_2, x_3) = (x_1, x_2 + 1, x_3)$

(d) $f_4 : \mathbb{R}^3 \to \mathbb{R}^3 \quad f_4(x_1, x_2, x_3) = (x_1 + x_2, x_2, x_1 - x_3)$

- Which are linear?
- For each linear transformation in the above list write the associated matrix.
- Which of the linear transformations in the above list are invertible?
- For each invertible linear transformation in the above list write the inverse transformation explicitly.

4. [25 points] (**ANSWER THIS QUESTION OR NUMBER 5**)

State whether the following statements are true or false. If true explain your answer, if false give an example for which the statement is false:

(a) Let A, B and C be three (2×2)-matrices.

$$AB = AC \implies B = C$$

(b) If the 3×3 matrix A has rank 3, then any linear system with coefficient matrix A has a unique solution.

(c) The vector $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ is orthogonal to the vector $\begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$

(d) For a $n \times m$ matrix A and 2 vectors $\overrightarrow{b_1}, \overrightarrow{b_2} \in \mathbb{R}^m$ the following statement holds

$$A \cdot \overrightarrow{x} = \overrightarrow{b_i} \text{ inconsistent system, for } i = 1, 2 \implies A \cdot \overrightarrow{x} = (\overrightarrow{b_1} + \overrightarrow{b_2}) \text{ inconsistent system}$$

5. [25 points] **(ANSWER THIS QUESTION OR NUMBER 4)**

State whether the following statements are true or false. If true explain your answer, if false give an example for which the statement is false:

(a) If A is a 2×2 matrix such that $A^2 = A \cdot A = 0$, then $A = 0$ (the zero matrix).

(b) If A, B and C are 2×2 invertible matrices, then $(A + B)C$ is invertible.

(c) $\vec{u}$, $\vec{v}$ orthogonal, unit vectors $\implies$ $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$ orthogonal vectors.

(d) The matrix $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ describes a reflection about a line in $\mathbb{R}^2$.

6. [20 points] **(BONUS: ANSWER THIS QUESTION TO GET EXTRA POINTS)**

Define the linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ (i.e. write the matrix A) satisfying the following requirements:

(a) $T_A(\overrightarrow{x}) = \overrightarrow{0}$ if (and only if) $\overrightarrow{x} = (c)\overrightarrow{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $(c \in \mathbb{R})$

(b) $T_A(x, y, z) = \{(x, y, z) \in \mathbb{R}^3 \mid x + z = 0\}$ for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \overrightarrow{b}$

Is such T_A invertible? Justify your answer.

Hint. $\overrightarrow{b}$ is orthogonal to the plane $x+z = 0$. Thus $T_A(\overrightarrow{x}) = \overrightarrow{x} - (\overrightarrow{x})^{\perp}$, where $(\overrightarrow{x})^{\perp}$ is the orthogonal projection of $\overrightarrow{x}$ along (the line through) $\overrightarrow{b}$.