201 LINEAR ALGEBRA, MIDTERM 1
March 8, 2011

NAME:

Section no:

TA:

. **There are 7 pages in the exam including this page.**

. **Write all your answers clearly. You have to show work to get points for your answers.**

. **Use of Calculators is not allowed during the exam.**

I agree to complete this exam without unauthorized assistance from any person, materials or device.

Signature:

Date:

1 (10)	
2 (10)	
3 (20)	
4 (10)	
Total (50)	

Duration: 50 mins

1. *10 points.* Solve the system

$$x + 2y + 3z = 1$$

$$4x + 5y + 9z = 1$$

$$7x + 8y + 15z = 1$$

Determine the rank of the coefficient matrix.

The augmented matrix $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 9 & 1 \\ 7 & 8 & 15 & 1 \end{pmatrix}$ row reduces to $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore the solution set is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t-1 \\ -t+1 \\ t \end{pmatrix}; t \in \mathbb{R}.$$

The rank of the coffecient matrix is 2.

2. *10 points.* True or false. Justify your answer.

(a) There exists a 2×2 matrix A such that $A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

$A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which is impossible.
FALSE.

(b) There is a 2×3 matrix A of rank 2 such that $A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

TRUE. $A = \begin{pmatrix} 2 & -2 & 1 \\ 3 & -3 & 5 \end{pmatrix}$.

3. *20 points.*

(a) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x+z \end{pmatrix}$$

Show that T is an invertible linear transformation. Compute the matrix for T^{-1}.

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The augmented matrix $\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$ row reduces to $\begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{pmatrix}$.

Therefore

$$T^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(b) Let $L = \{t\vec{e}_1 | t \in \mathbb{R}\}$ be the line in $\mathbb{R}^3$ spanned by $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Find the matrix of the transformation $\text{proj}_L \circ T : \mathbb{R}^3 \to \mathbb{R}^3$, where T is the linear transformation in part (a) and $\text{proj}_L \circ T(\vec{x}) = \text{proj}_L(T(\vec{x}))$.

$\text{proj}_L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Therefore,

$$\text{proj}_L \circ T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(c) Describe the kernel and image of the transformation in part (b) in terms of a span of vectors in $\mathbb{R}^3$. Use as few vectors as possible.

$$\mathrm{Im}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\} = \mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\} = L.$$

$$\mathrm{Ker}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ s \end{pmatrix}, s, t \in \mathbb{R} = \mathrm{span}\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}.$$

(d) Consider the column vectors of the matrix in part (b). Are they linearly independent? If not, find out all possible linear dependency relations among them.

The columns are linearly dependent since the matrix has a non-zero kernel. The vectors in the kernel are exactly the dependency relations;

$$t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

4. *10 points.* True or False. Justify your answer.

(a) There exists an invertible $n \times n$ matrix with two identical columns.

Suppose columns C_i and C_j are identical, $1 \leq i, j \leq n$, then there is a linear depencency relation among the columns;

$$C_i - C_j = \vec{0}.$$

Therefore the matrix has a non-zero kernel. FALSE.

(b) There exists a 2×3 matrix A and a 3×2 matrix B such that $BA = I_3$.

The maximum rank of A is 2. Therefore by rank-nullity theorem A must have non-zero kernel. $\mathrm{Ker}A \subset \mathrm{Ker}(BA)$ implies BA must have a non-zero kernel as well. FALSE.