Math 201
Spring 2014
Midterm 1
02/26/14
Lecturer: Jesus Martinez Garcia
Time Limit: 50 minutes

Name (Print): ____________________

Teaching Assistant ____________________

This exam contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a theorem of lemma you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (25 points) **Solving linear systems.** Consider the following linear system:

$$\left\{\begin{array}{llll} x & -y & +(k^2-1)z & = 1-k \\ 2x & -y & +(4k^2-4)z & = 4-k \\ -3x & +4y & +(-2k^2+2)z & = -2+4k \end{array}\right\},$$

where $k \in \mathbb{R}$.

(a) (5 points) Write the augmented matrix $(A|\overrightarrow{b})$ of the linear system.

Solution

The augmented matrix is

$$(A|\overrightarrow{b}) = \left(\begin{array}{lll|c} 1 & -1 & (k^2-1) & 1-k \\ 2 & -1 & 4k^2-4 & 4-k \\ -3 & 4 & -2k^2+2 & -2+4k \end{array}\right).$$

(b) (15 points) Consider k is fixed. Find $\mathrm{rref}(A|\overrightarrow{b})$.

Solution

We perform Gauss–Jordan elimination:

$$\left(\begin{array}{lll|c} 1 & -1 & (k^2-1) & 1-k \\ 2 & -1 & 4k^2-4 & 4-k \\ -3 & 4 & -2k^2+2 & -2+4k \end{array}\right) \longrightarrow \left(\begin{array}{lll|c} 1 & -1 & (k^2-1) & 1-k \\ 2 & -1 & 4(k^2-1) & 4-k \\ -3 & 4 & -2(k^2-1) & -2+4k \end{array}\right) \xrightarrow[(III):+3(I)]{(II):-2(I)}$$

$$\left(\begin{array}{lll|c} 1 & -1 & k^2-1 & 1-k \\ 0 & 1 & 2(k^2-1) & 2+k \\ 0 & 1 & k^2-1 & 1+k \end{array}\right) \xrightarrow[(III):+(II)]{(I):+(II)} \left(\begin{array}{lll|c} 1 & 0 & 3(k^2-1) & 3 \\ 0 & 1 & 2(k^2-1) & 2+k \\ 0 & 0 & -(k^2-1) & -1 \end{array}\right) \xrightarrow[(III):\div 3]{}$$

$$\left(\begin{array}{lll|c} 1 & 0 & 3(k^2-1) & 3 \\ 0 & 1 & 2(k^2-1) & 2+k \\ 0 & 0 & -(k^2-1) & -1 \end{array}\right) \xrightarrow[(II):-(III)]{(I):-3(III)} \left(\begin{array}{lll|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & (k^2-1) & 1 \end{array}\right).$$

Now, if $k \neq \pm 1$, then

$$\mathrm{rref}(A|\overrightarrow{b}) = \left(\begin{array}{lll|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & & \frac{1}{(k^2-1)} \end{array}\right).$$

If $k = \pm 1$, then

$$\left(\begin{array}{lll|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & 0 & 1 \end{array}\right).$$

(c) (5 points) Analyse whether the linear system has solutions or not according to the values of k. For each k such that solutions exist, find all solutions.

Solution

If $k = \pm 1$, then $\mathrm{rk}(A) = 2 \neq 3 = \mathrm{rk}(A|\overrightarrow{b})$, and by the rank-solutions theorem, the system has no solutions. If $k \neq \pm 1$, then $\mathrm{rk}(A|\overrightarrow{b}) = \mathrm{rk}(A) = 3$, and by the rank-solutions theorem the system has a unique solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ k \\ \frac{1}{k^2-1} \end{pmatrix}.$$

2. (25 points) **Inverses and transformations on the plane.** Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) (10 points) Decide if A and B are invertible. If they are, find the inverses. If they are not, say why not.

Solution: Recall that a matrix M is invertible, if and only if $\mathrm{rk}(M) = 2$, if and only if $\mathrm{rref}(M) = I_2$.

Therefore A is not invertible, since its reduced row echelon form is

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

The matrix B is indeed invertible. We find the inverse by means of finding the reduced row echelon form of the following matrix:

$$\left(\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \end{array}\right) \xrightarrow[\times(\sqrt{2})]{\times(\sqrt{2})} \left(\begin{array}{cc|cc} 1 & 1 & \sqrt{2} & 0 \\ -1 & 1 & 0 & -\sqrt{2} \end{array}\right) \xrightarrow[+(I)]{}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & \sqrt{2} & 0 \\ 0 & 2 & \sqrt{2} & -\sqrt{2} \end{array}\right) \xrightarrow[\div 2]{} \left(\begin{array}{cc|cc} 1 & 1 & \sqrt{2} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}\right) \xrightarrow{-(I)} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}\right).$$

Observe, in particular that $B^{-1} = B$.

(b) (5 points) The matrices A and B define certain well known linear transformations T_A and T_B on the plane. Say which transformations they are (i.e. give their name and say why).

Solution

For A we have:

$$A = \begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix} = \begin{pmatrix} (\frac{1}{\sqrt{2}})^2 & (\frac{1}{\sqrt{2}})^2 \\ (\frac{1}{\sqrt{2}})^2 & (\frac{1}{\sqrt{2}})^2 \end{pmatrix}.$$

Therefore A is the matrix of a projective transformation over the line generated by the vector $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(1, 1)$. For B notice that $B = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, where $a^2 + b^2 = 1$, so it gives the matrix of a reflection.

(c) (10 points) A and B fix all points in some lines L_A and L_B respectively, i.e $A\vec{x} = \vec{x}$ $\forall \vec{x} \in L_A$ and $B\vec{x} = \vec{x} \forall \vec{x} \in L_B$. Give the equations of L_A and L_B.

Solution

Since A is a projection over the line generated by $(1, 1)$, the points in this line are fixed by A and the equation of the line is $x - y = 0$, since $\begin{pmatrix} x \\ y \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For B we look for $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $B \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x + y \\ x - y \end{pmatrix}.$$

Therefore we have the homogeneous linear system of equations

$$\left\{ \begin{array}{lll} (1-\frac{1}{\sqrt{2}})x & -\frac{1}{\sqrt{2}}y & =0 \\ -\frac{1}{\sqrt{2}}x & +(1+\frac{1}{\sqrt{2}})y & =0 \end{array} \right\},$$

which is equivalent to

$$\left\{ \begin{array}{llll} (I): & (\sqrt{2}-1)x & -y & =0 \\ (II): & -x & +(1+\sqrt{2})y & =0 \end{array} \right\}.$$

Since

$$\alpha := \frac{-1}{\sqrt{2}-1} = -\frac{1+\sqrt{2}}{2-1}$$

multiplying (I) by α, we obtain (II) and (II) is therefore redundant. The line defined by B is the one with equation $y = (\sqrt{2}-1)x$.

3. (25 points) **Image, kernel and bases.** Let

$$T(\overrightarrow{x}) = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 2 & 2 & 4 \\ 3 & 6 & 6 & 0 & 3 \end{pmatrix} \cdot \overrightarrow{x}$$

where $\overrightarrow{x} \in \mathbb{R}^5$.

(a) (10 points) Find a basis for the image of T. Justify why the vectors you provide are linearly independent and span $\mathrm{Im}(T)$.

Solution

The image is spanned by the columns of the matrix of T. Call this matrix A and its columns $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_5 \in \mathbb{R}^4$. Hence $\mathrm{Im}(A) = \langle \overrightarrow{v}_1, \ldots, v_5 \rangle$. We need to find the redundant vectors of A. There are several ways to do this, but we will choose the one that uses the reduced row echelon form. We apply Gauss–Jordan elimination to A:

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 2 & 2 & 4 \\ 3 & 6 & 6 & 0 & 3 \end{pmatrix} \xrightarrow[\substack{(III):-2(I) \\ (IV):-3(I)}]{(II):-(I)} \begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 2 & -4 & 2 & 2 \\ 0 & 0 & -3 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{(III):-2(II) \\ (IV):\div(-3)}]{(I):-2(II)} \begin{pmatrix} 1 & 0 & 7 & -2 & -1 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{(I):-7(IV) \\ (II):+2(IV)}} \begin{pmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{Swap } (II) \text{ and } (III)]{} \begin{pmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$= \mathrm{rref}(A) = \begin{pmatrix} | & | & | & | & | \\ \overrightarrow{w}_1 & \overrightarrow{w}_2 & \overrightarrow{w}_3 & \overrightarrow{w}_4 & \overrightarrow{w}_5 \\ | & | & | & | & | \end{pmatrix}.$$

Observe that the relations among the elements of B give us relations among the elements of A:

$$\overrightarrow{w}_4 = \overrightarrow{w}_2 - 2\overrightarrow{w}_1 \iff \overrightarrow{v}_4 = \overrightarrow{v}_2 - 2\overrightarrow{v}_1.$$

$$\overrightarrow{w}_5 = \overrightarrow{w}_2 - \overrightarrow{w}_1 \iff \overrightarrow{v}_5 = \overrightarrow{v}_2 - 2\overrightarrow{v}_1.$$

Therefore $\overrightarrow{v}_4$ and $\overrightarrow{v}_5$ are redundant. The vectors $\overrightarrow{w}_1, \overrightarrow{w}_2, \overrightarrow{w}_3$ are linearly independent, therefore $\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3$ are linearly independent too, and since they span $\mathrm{Im}(A)$, the set $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ is a basis of $\mathrm{Im}(A)$.

(b) (15 points) Find a basis for the kernel of T. Justify why the vectors you provide are linearly independent and span $\mathrm{Ker}(T)$.

Solution

From the reduced row echelon form of A, we deduce that the vector $\overrightarrow{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathrm{Ker}(A)$

if and only if

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2r+s \\ -r-s \\ 0 \\ r \\ s \end{pmatrix} = r \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = r\overrightarrow{a} + s\overrightarrow{b}.$$

Therefore $\langle \overline{a}, \overline{b} \rangle = \mathrm{Ker}(A)$. The vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are clearly linearly independent since they are not multiples of each other. Moreover, by the rank-nullity theorem we have:

$$\dim(\mathrm{Ker}(A)) = 5 - \dim(\mathrm{Im}(A)) = 5 - 3 = 2.$$

Therefore $\{\overrightarrow{a}, \overrightarrow{b}\}$ is a basis of $\mathrm{Ker}(A)$.

4. (25 points) **Properties of linear transformations.**

(a) (15 points) Let $\mathrm{Mat}_{3\times 2}(\mathbb{R})$ be the space of all 3×2 matrices. We can identify $\mathrm{Mat}_{3\times 2}(\mathbb{R})$ with $\mathbb{R}^6$ in the following way:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_2 \end{pmatrix} \xleftrightarrow{1:1} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, \text{ or more concisely, in vector form: } \begin{pmatrix} | & | \\ \vec{a} & \vec{b} \\ | & | \end{pmatrix} \xleftrightarrow{1:1} \begin{pmatrix} | \\ \vec{a} \\ | \\ | \\ \vec{b} \\ | \end{pmatrix}.$$

Since the sum of matrices and the product of a matrix by a scalar in $\mathrm{Mat}_{3\times 2}(\mathbb{R})$ corresponds to that of $\mathbb{R}^6$, we can think of $\mathrm{Mat}_{3\times 2}(\mathbb{R})$ as a vector space. Show that the following map is a linear transformation:

$$f\colon \mathrm{Mat}_{3\times 2}(\mathbb{R}) \cong \mathbb{R}^6 \longrightarrow \mathbb{R}^3, \qquad f\left(\begin{pmatrix} | & | \\ \vec{a} & \vec{b} \\ | & | \end{pmatrix}\right) = 2\vec{a} - \vec{b}.$$

Solution

The map f is linear if it satisfies

(i) $f(\vec{u}_1 + \vec{u}_2) = f(\vec{u}_1) + f(\vec{u}_2)$ for all $\vec{u}_1, \vec{u}_2 \in \mathrm{Mat}_{3\times 2}(\mathbb{R})$,

(i) $f(k\vec{u}_1) = kf(\vec{u}_1)$ for all $\vec{u}_1 \in \mathrm{Mat}_{3\times 2}(\mathbb{R})$ and $k \in \mathbb{R}$.

Indeed, this is the case. We let $\vec{u}_1 = \begin{pmatrix} | & | \\ \vec{a}_1 & \vec{b}_1 \\ | & | \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} | & | \\ \vec{a}_2 & \vec{b}_2 \\ | & | \end{pmatrix}$ and $k \in \mathbb{R}$. Then

(i)

$$f(\vec{u}_1 + \vec{u}_2) = f\left(\begin{pmatrix} | & | \\ \vec{a}_1 & \vec{b}_1 \\ | & | \end{pmatrix} + \begin{pmatrix} | & | \\ \vec{a}_2 & \vec{b}_2 \\ | & | \end{pmatrix}\right) = f\left(\begin{pmatrix} | & | \\ \vec{a}_1 + \vec{a}_2 & \vec{b}_1 + \vec{b}_2 \\ | & | \end{pmatrix}\right) =$$

$$= 2(\vec{a}_1 + \vec{a}_2) - (\vec{b}_1 + \vec{b}_2) = (2\vec{a}_1 - \vec{b}_1) + (2\vec{a}_2 - \vec{b}_2) = f(\vec{u}_1) + f(\vec{u}_2).$$

(ii)

$$f(k\cdot\vec{u}_1) = f\left(k\cdot\begin{pmatrix} | & | \\ \vec{a}_1 & \vec{b}_1 \\ | & | \end{pmatrix}\right) = f\left(\begin{pmatrix} | & | \\ k\vec{a}_1 & k\vec{b}_1 \\ | & | \end{pmatrix}\right) =$$

$$= 2(k\vec{a}_1) - (k\vec{b}_1) = k(2\vec{a}_1 - \vec{b}_1) = kf(\vec{u}_1).$$

Therefore, f is a linear transformation.

(b) (10 points) Consider the following statements. If they are true, provide a proof. If they are false, provide a counter-example.

(i) Given square matrices A and B of the same size, if $A\cdot B = 0$, then $A = 0$ or $B = 0$.

(ii) Let A be a square matrix. Then $\mathrm{Ker}(A) \subseteq \mathrm{Ker}(A^2)$.

Solution

(i) False. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $AB = 0$.

(ii) True. Let $\overrightarrow{x} \in \mathrm{Ker}(A)$. Then $A\overrightarrow{x} = \overrightarrow{0}$, so

$$A^2\overrightarrow{x} = A(A\overrightarrow{x}) = A\overrightarrow{0} = \overrightarrow{0}.$$

Therefore $\overrightarrow{x} \in \mathrm{Ker}(A^2)$.