LINEAR ALGEBRA (MATH 110.201)

MIDTERM I - 26 FEBRUARY 2016

Name: ______________________________

Section number/TA: ______________________________

Instructions:

(1) Do not open this packet until instructed to do so.
(2) This midterm should be completed in **50 minutes**.
(3) Notes, the textbook, and digital devices **are not permitted**.
(4) Discussion or collaboration is **not permitted**.
(5) All solutions must be written on the pages of this booklet.
(6) Justify your answers, and write clearly; **points will be subtracted otherwise**.
(7) Once you submit your exam, you will not be allowed to modify it.
(8) By submitting this exam, you are agreeing to the above terms. Cheating or refusing to stop writing when time is called may result in an automatic failure.

Exercise	Points	Your score
1	12	12
2	12	12
3	12	12
4	12	12
5	16	16
6	12	12
7	12	12
8	12	12
Total	100	100

Exercise 1. *Consider the following system of equations, where λ is a real number:*

$$\begin{aligned} 1W + 2X + 3Y + 4Z &= 1 \\ 5W + 6X + 7Y + 8Z &= \lambda \\ 9W + 10X + 11Y + 12Z &= \lambda^2 \end{aligned}$$

(1) *(4 points) Write down the augmented matrix of this system.*

(2) *(8 points) For which real numbers λ does this system have at least one solution? When it has at least one solution, describe all solutions.*

Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 8 & \lambda \\ 9 & 10 & 11 & 12 & \lambda^2 \end{array}\right] \begin{array}{l} \\ -5\mathrm{I} \\ -9\mathrm{I} \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & -4 & -8 & -12 & \lambda-5 \\ 0 & -8 & -16 & -24 & \lambda^2-9 \end{array}\right] \div(-4) \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 & -\frac{\lambda-5}{4} \\ 0 & -8 & -16 & -24 & \lambda^2-9 \end{array}\right] \begin{array}{l} -2\mathrm{II} \\ \\ +8\mathrm{II} \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & \frac{\lambda-3}{2} \\ 0 & 1 & 2 & 3 & -\frac{\lambda-5}{4} \\ 0 & 0 & 0 & 0 & \lambda^2-2\lambda+1 \end{array}\right]$$

The system has at least one solution if and only if $\lambda^2-2\lambda+1 = (\lambda-1)^2 = 0$, i.e. if and only if $\lambda = 1$.

When $\lambda = 1$, the solutions of the system are

$$W = Y + 2Z - 1$$

$$X = -2Y - 3Z + 4 \quad \text{i.e.}$$

$$\begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} s+2t-1 \\ -2s-3t+4 \\ s \\ t \end{bmatrix}$$ where s and t are arbitrary numbers.

Exercise 2. *Consider the following vectors in* $\mathbb{R}^3$*:*

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

(1) *(4 points) Calculate* $||\vec{x}||$ *and* $||\vec{y}||$.
(2) *(4 points) Calculate the distance between* $\vec{x}$ *and* $\vec{y}$.
(3) *(4 points) Are* $\vec{x}$ *and* $\vec{y}$ *perpendicular to each other?*

Solution:

(1) $||\vec{x}|| = \sqrt{1+4+9} = \sqrt{14}$

$||\vec{y}|| = \sqrt{1+4+9} = \sqrt{14}$

(2) $||\vec{x} - \vec{y}|| = \left|\left| \begin{bmatrix} -3 \\ -1 \\ -4 \end{bmatrix} \right|\right| = \sqrt{9+1+16} = \sqrt{26}$

(3) $\vec{x} \cdot \vec{y} = -2 + 6 - 3 = 1 \neq 0$, so $\vec{x}$ and $\vec{y}$ are not perpendicular to each other.

Exercise 3. *(12 points) Find all vectors in $\mathbb{R}^3$ which are simultaneously perpendicular to both of the following vectors:*

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \textit{and} \qquad \vec{w} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \perp \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 \Rightarrow$$

$$a + 2b + 3c = 0 \qquad (1)$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \perp \vec{w} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = 0 \Rightarrow$$

$$2a + 5b - c = 0. \qquad (2)$$

Solve (1) and (2):

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & -1 & 0 \end{array}\right] \begin{array}{l} \\ -2\,\mathrm{I} \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -7 & 0 \end{array}\right] \begin{array}{l} -2\,\mathrm{II} \\ \\ \end{array} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 17 & 0 \\ 0 & 1 & -7 & 0 \end{array}\right].$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -17s \\ 7s \\ s \end{bmatrix} \quad \text{where } s \text{ is arbitrary.}$$

So the vectors in $\mathbb{R}^3$ that are simultaneously perpendicular to $\vec{v}$ and $\vec{w}$ are

$$\begin{bmatrix} -17s \\ 7s \\ s \end{bmatrix} \quad \text{where } s \text{ can be any number.}$$

Exercise 4. *Decide whether the following functions are linear transformations. If they are linear transformations, write down their matrices. If not, justify why not.*

(1) *(6 points)* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+1 \\ 3y-1 \end{bmatrix}$$

(2) *(6 points)* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *given by*

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-3y \\ 7y+2x \\ 3x-y \end{bmatrix}$$

Solution:

(1) Not a linear transformation because of the constant terms in the two component functions of T.

(2) Yes, T is a linear transformation.

$$\begin{bmatrix} 2 & -3 \\ 2 & 7 \\ 3 & -1 \end{bmatrix}.$$

Exercise 5. *Consider* $\mathrm{Rot}_{\pi/4} : \mathbb{R}^2 \to \mathbb{R}^2$ *counter-clockwise rotation by* 45 *degrees, and* $\mathrm{Ref}_{\mathscr{L}}$ *reflection over the line* $\mathscr{L}$ *passing through the origin and* $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$.

(1) *(8 points) Write down the matrices of* $\mathrm{Rot}_{\pi/4}$ *and* $\mathrm{Ref}_{\mathscr{L}}$.

(2) *(4 points) Write down the matrix of* $\mathrm{Rot}_{\pi/4} \circ \mathrm{Ref}_{\mathscr{L}}$.

(3) *(4 points) Write down the matrix of* $\mathrm{Ref}_{\mathscr{L}} \circ \mathrm{Rot}_{\pi/4}$.

Solution:

(1) $\mathrm{Rot}_{\frac{\pi}{4}} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$

The matrix for the orthogonal projection onto the line $\mathcal{L}$ is $P = \frac{1}{1+4}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}.$

so $\mathrm{Ref}_{\mathcal{L}} = 2P - I_2 = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

(2) $\mathrm{Rot}_{\frac{\pi}{4}} \circ \mathrm{Ref}_{\mathcal{L}} : \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

$= \begin{bmatrix} \frac{7}{10}\sqrt{2} & \frac{1}{10}\sqrt{2} \\ \frac{1}{10}\sqrt{2} & \frac{7}{10}\sqrt{2} \end{bmatrix}$

(3) $\mathrm{Ref}_{\mathcal{L}} \circ \mathrm{Rot}_{\frac{\pi}{4}} : \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{10}\sqrt{2} & \frac{7}{10}\sqrt{2} \\ \frac{7}{10}\sqrt{2} & -\frac{1}{10}\sqrt{2} \end{bmatrix}$

Exercise 6. *Consider the following matrix:*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(1) *(8 points) Find* **two** *different* 3×2 *matrices* C *with the property that* $AC = I_2$.
(2) *(4 points) Is it possible to find a* 3×2 *matrix* B *with the property that* $BA = I_3$?

Solution:

(1) Suppose $C = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. $AC = \begin{bmatrix} a+c & b+d \\ a+e & b+f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Therefore $a+c=1$, $b+d=0$, $a+e=0$, $b+f=1$. e.g. $a=0$, $c=1$, $b=0$, $d=0$, $e=0$, $f=1$, then

$C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$; or $a=1$, $c=0$, $b=0$, $d=0$, $e=-1$, $f=1$, then

$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$.

(2) ~~No, it's not possible~~

~~I_3 represents the identity matri~~

No, it's not possible.

I_3 represents the identity transformation: $I_3 \vec{x} = \vec{x}$,

so $\ker(I_3) = \{\vec{0}\}$.

We claim that $\ker(BA)$ contains $\ker(A)$. Indeed,

if $\vec{x} \in \ker(A)$, then $A\vec{x} = \vec{0} \Rightarrow BA\vec{x} = B(A\vec{x}) = \vec{0}$

$\Rightarrow \vec{x} \in \ker(BA)$. Hence, $\ker(BA)$ contains $\ker(A)$.

By the rank-nullity theorem, $\dim \ker(A) + \operatorname{rank}(A) = 3$.

However, A has two rows, so $\operatorname{rank}(A) \leq 2$. Therefore,

$\dim \ker(A) \geq 1$.

Since $\ker(BA)$ contains $\ker(A)$, it follows that $\dim \ker(BA) \geq 1$.

Thus, $\ker(BA) \neq \{\vec{0}\}$. This shows that it is impossible to find a 3×2 matrix B such that $BA = I_3$.

Exercise 7. *(12 points) Is the following matrix invertible?*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

If so, write down A^{-1}.

Solution:

$$\begin{array}{cc} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \\ -2\mathrm{I} \\ -3\mathrm{I} \\ -4\mathrm{I} \end{array} \end{array} \to \begin{array}{cc} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \\ \\ -2\mathrm{II} \\ -3\mathrm{II} \end{array} \end{array}$$

$$\to \begin{array}{cc} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 5 & -3 & 0 & 1 \end{array}\right] & \begin{array}{l} \\ \\ \\ -2\mathrm{III} \end{array} \end{array} \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & -2 & 1 \end{array}\right]$$

The left half of the last matrix is I_4, so A is invertible and

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 3 & 1 & -2 & 1 \end{bmatrix}.$$

Exercise 8. *(12 points) Consider the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Is the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *in* $\mathrm{Im}(A)$?

Solution: $\mathrm{im}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$.

$$c_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{array}\right] \begin{array}{l} \\ -4\mathrm{I} \\ -7\mathrm{I} \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{array}\right] \div(-3) \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -6 & -12 & -6 \end{array}\right] \begin{array}{l} -2\mathrm{II} \\ \\ +6\mathrm{II} \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has a solution for c_1, c_2, c_3, so $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is $\mathrm{im}(A)$.