Math 201
Fall 2014
Midterm 2
12/03/14
Lecturer: Jesse Gell-Redman
Time Limit: 50 minutes

Name (Print): ____________________

Teaching Assistant ____________________

This exam contains 12 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Follow the instructions closely**. For example, if you are asked to justify your answers, then do so in a brief and coherent way.
- **Points will be taken off for incorrect statements, even if correct ones are present.** Be careful about what you include in your answers. If they contain both the correct answers and incorrect or nonsense statements, points will be taken off.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Good luck!! Do not write in the table to the right.

1. (20 points) **Linear transformations:** show your work. Consider the space of 2×2 matrices

$$Mat_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

(Your book calls this $\mathbb{R}^{2\times 2}$.)

(a) (5 points) Let $T\colon Mat_2 \longrightarrow Mat_2$ be a map. Write the properties which T must satisfy to be *linear.*

(b) (5 points) Define the basis $\mathcal{B}$ of Mat_2 by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \tag{1}$$

Write the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}$$

in the $\mathcal{B}$ basis, i.e. find $(A)_{\mathcal{B}}$.

(c) (10 points) Now assume that T is defined by

$$T(M) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} M,$$

and let $\mathcal{B}$ be the same basis as in the previous part, i.e. the basis $\mathcal{B}$ in equation (1). Write the transformation T in the basis $\mathcal{B}$; that is, find a matrix A so that

$$A(M)_{\mathcal{B}} = (T(M))_{\mathcal{B}}.$$

2. (20 points) **Gram-Schmidt and QR factorization**

   (a) (10 points) Apply the Gram-Schmidt process to the vectors

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 6 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 13 \end{pmatrix},$$

   to obtain orthonormal vectors $\vec{u}_1$ and $\vec{u}_2$.

Consider the matrix factorization

$$\begin{pmatrix} 1 & 1 \\ 1 & 9 \\ 1 & -5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 10 \end{pmatrix}$$

Assuming this is the QR factorization (it is!) do the following

(b) (3 points) Without doing computations, find an orthonormal basis of

$$V = \text{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ -5 \\ 3 \end{pmatrix} \right)$$

(c) (7 points) Find the orthogonal projection of $100\vec{e}_1$ onto the subspace V defined in part b). (Write out *all* of the components; do not merely express as a linear combination of other vectors.)

3. (20 points) **Orthogonality, Least squares**

   (a) (3 points) Complete the definition: an $m \times n$ matrix A is orthogonal if...

   (b) (3 points) Show that the rotation matrix, $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, is orthogonal.

   .

   (c) (4 points) Suppose that A and B are two $n \times n$ orthogonal matrices. Is the product AB also orthogonal? Justify your answer.

(d) (10 points) Find the least squares solution $\vec{x}^*$ of the system

$$A\vec{x} = \vec{b} \text{ where } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

4. (20 points) **Inner products, determinants:**

   (a) (6 points) Consider the inner product on the space $C(-1,1)$ of continuous functions on the interval $[-1,1]$,

   $$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt.$$

   Consider the subspace $Poly_1 = \text{span}\{1, t\}$ of first order (linear) polynomials, i.e. functions for the form $a + bt$ for real scalars a, b. Find an orthonormal basis for $Poly_1$.

(b) (6 points) Let $Poly_2$ denote the space of second order polynomials

$$Poly_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}.$$

Consider the linear map $T\colon Poly_2 \longrightarrow Poly_2$ defined for a polynomial $f(x)$ by

$$T(f) = f'' + 2f$$

where f'' denotes the second derivative. Calculate $\det(T)$. Based on this, what can you say about $\ker(T)$.

(c) (4 points) Find the determinant of the following matrix and say whether or not it is invertible.

$$\begin{pmatrix} 1 & 4 & 0 \\ 2 & 0 & 5 \\ 3 & 0 & 0 \end{pmatrix}.$$

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(d) (4 points) Suppose that A is a 2×2, and that $\text{Trace}(A) = 0$, while $\det(A) = -4$. Find the eigenvalues of A. (The trace $\text{Trace}(A)$ is the sum of the diagonal entries of A, but you may use any facts you know about it and the determinant.)

5. (20 points) **Eigenvalues and eigenvectors:**

   (a) (6 points) Complete the three equivalent definitions of diagonalizability. An $n \times n$ matrix A is diagonalizable if

   - there is a basis of $\mathbb{R}^n$ consisting of...

     A is similar to a diagonal matrix, meaning...

   - All of the eigenvalues of A are real and for each one the algebraic multiplicity is ...

   .

   (b) (2 points) Show that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is NOT diagonalizable. (You may just draw a picture)

(c) (12 points) Compute the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

For each eigenvalue, what are the algebraic and geometric multiplicities. Is the matrix diagonalizable?