Math 201
Fall 2014
Midterm 2
12/03/14
Lecturer: Jesse Gell-Redman
Time Limit: 50 minutes

Name (Print): ____________________

Teaching Assistant ____________________

This exam contains 12 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Follow the instructions closely**. For example, if you are asked to justify your answers, then do so in a brief and coherent way.
- **Points will be taken off for incorrect statements, even if correct ones are present.** Be careful about what you include in your answers. If they contain both the correct answers and incorrect or nonsense statements, points will be taken off.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Good luck!! Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) **Linear transformations:** show your work. Consider the space of 2×2 matrices

$$Mat_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

(Your book calls this $\mathbb{R}^{2\times 2}$.)

(a) (5 points) Let $T\colon Mat_2 \longrightarrow Mat_2$ be a map. Write the properties which T must satisfy to be *linear.*

Answer: T must satisfy that for any matrices $A, B \in Mat_2$, that $T(A+B) = T(A)+T(B)$, and that for any scalar $k \in \mathbb{R}$ that $T(kA) = kT(A)$.

(b) (5 points) Define the basis $\mathcal{B}$ of Mat_2 by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \tag{1}$$

Write the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}$$

in the $\mathcal{B}$ basis, i.e. find $(A)_{\mathcal{B}}$.

Answer: Note that

$$\begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Thus,

$$(A)_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

(c) (10 points) Now assume that T is defined by

$$T(M) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} M,$$

and let $\mathcal{B}$ be the same basis as in the previous part, i.e. the basis $\mathcal{B}$ in equation (1). Write the transformation T in the basis $\mathcal{B}$; that is, find a matrix A so that

$$A(M)_{\mathcal{B}} = (T(M))_{\mathcal{B}}.$$

Answer: Recall that

$$A = \begin{pmatrix} | & | & | & | \\ (T(f_1))_{\mathcal{B}} & (T(f_2))_{\mathcal{B}} & (T(f_3))_{\mathcal{B}} & (T(f_4))_{\mathcal{B}} \\ | & | & | & | \end{pmatrix},$$

where $f_1, \ldots, f_4$ are the four basis elements in $\mathcal{B}$ in the order they appear above. We check that

$$(T(f_1))_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} f_1\right)_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}\right)_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$(T(f_2))_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} f_2\right)_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right)_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\right)_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

$$(T(f_3))_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} f_3\right)_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)_{\mathcal{B}} = \left(\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}\right)_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \\ 4 \\ 0 \end{pmatrix}$$

$$(T(f_4))_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} f_4\right)_{\mathcal{B}} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right)_{\mathcal{B}} = \left(\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\right)_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}.$$

Thus A is given by

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}$$

2. (20 points) **Gram-Schmidt and QR factorization**

(a) (10 points) Apply the Gram-Schmidt process to the vectors

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 6 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 13 \end{pmatrix},$$

to obtain orthonormal vectors $\vec{u}_1$ and $\vec{u}_2$.

Answer:

$$\|\vec{v}_1\|^2 = 2^2 + 3^2 + 6^2 = 49 = 7^2,$$

so

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} v_1 = \frac{1}{7}\vec{v}_1 = (1/7)\begin{pmatrix} 2 \\ 3 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2/7 \\ 3/7 \\ 0 \\ 6/7 \end{pmatrix}$$

Then $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$, but

$$\vec{v}_2 \cdot \vec{u}_1 = (1/7)\begin{pmatrix} 4 \\ 4 \\ 2 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \\ 6 \end{pmatrix} = \frac{1}{7}(8 + 12 + 78) = 14,$$

so

$$\vec{v}_2^{\perp} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 13 \end{pmatrix} - 14\begin{pmatrix} 2/7 \\ 3/7 \\ 0 \\ 6/7 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \\ 1 \end{pmatrix}.$$

Finally, since $\|\vec{v}_2^{\perp}\| = \sqrt{(4+4+1)} = 3$, we have

$$\vec{u}_2 = \begin{pmatrix} 0 \\ -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

This gives $\vec{u}_1$ and $\vec{u}_2$.

Consider the matrix factorization

$$\begin{pmatrix} 1 & 1 \\ 1 & 9 \\ 1 & -5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 10 \end{pmatrix}$$

Assuming this is the QR factorization (it is!) do the following

(b) (3 points) Without doing computations, find an orthonormal basis of

$$V = \text{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ -5 \\ 3 \end{pmatrix} \right)$$

Answer: The columns of Q (the left matrix on the right hand side) form an orthonormal basis for V, thus

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad \text{and } \vec{u}_2 = \begin{pmatrix} -1/10 \\ 7/10 \\ -7/10 \\ 1/10 \end{pmatrix}$$

(c) (7 points) Find the orthogonal projection of $100\vec{e}_1$ onto the subspace V defined in part b). (Write out *all* of the components; do not merely express as a linear combination of other vectors.)

Answer: There are several ways to proceed. We do so by recalling that the orthogonal projection onto V of a vector $\vec{v}$ is given by

$$\text{Proj}_V(\vec{v}) = (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\vec{u}_2,$$

where $\vec{u}_1, \vec{u}_2$ are the orthonormal basis from the previous part. Computing gives

$$\text{Proj}_V\left(100\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 25 \\ 25 \\ 25 \\ 25 \end{pmatrix} - \begin{pmatrix} -1 \\ 7 \\ -7 \\ 1 \end{pmatrix} = \begin{pmatrix} 26 \\ 18 \\ 32 \\ 24 \end{pmatrix}.$$

3. (20 points) **Orthogonality, Least squares**

(a) (3 points) Complete the definition: an $m \times n$ matrix A is orthogonal if...

Answer 1: The columns form an orthonormal set.

Answer 2: $A^T A = I_m$.

Answer 3: $\|A\vec{v}\| = \|\vec{v}\|$ for all $\vec{v}$. (Any of these answers is fine.)

(b) (3 points) Show that the rotation matrix, $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, is orthogonal.

Answer:

$$M^T M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = I_2$$

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(c) (4 points) Suppose that A and B are two $n \times n$ orthogonal matrices. Is the product AB also orthogonal? Justify your answer.

Answer: Yes.

$$(AB)^T(AB) = B^T A^T AB = B^T(A^T A)B = B^T I_n B = B^T B = I_n.$$

(d) (10 points) Find the least squares solution $\vec{x}^*$ of the system

$$A\vec{x} = \vec{b} \text{ where } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

Answer: Recall that

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}.$$

Here

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Thus

$$(A^T A)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and so

$$\vec{x}^* = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

4. (20 points) **Inner products, determinants:**

   (a) (6 points) Consider the inner product on the space $C(-1,1)$ of continuous functions on the interval $[-1,1]$,

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt.$$

Consider the subspace $Poly_1 = \text{span}\{1, t\}$ of first order (linear) polynomials, i.e. functions for the form $a + bt$ for real scalars a, b. Find an orthonormal basis for $Poly_1$.

Answer: We preform Gram-Schmidt to the basis $f_1 = 1, f_2(t) = t$. Note that

$$\|f_1\|^2 = \langle f_1, f_1\rangle = \int_{-1}^{1} f_1^2 dt = 2,$$

so we set

$$g_1 = \frac{1}{\|f_1\|} f_1 = \frac{1}{\sqrt{2}}.$$

Then note that $\langle f_2, f_1\rangle = \int_{-1}^{1} t dt = 0$, so f_2 and f_1 are already orthogonal. Thus, we must only normalize f_2, i.e. define $g_2 = \frac{1}{\|f_2\|} f_2$. Computing

$$\|f_2\|^2 = \int_{-1}^{1} t^2 dt = 2/3,$$

so

$$g_2 = \sqrt{\frac{3}{2}} t,$$

and

$$g_1, g_2 \text{ form an orthonormal basis of } Poly_1.$$

(b) (6 points) Let $Poly_2$ denote the space of second order polynomials

$$Poly_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}.$$

Consider the linear map $T\colon Poly_2 \longrightarrow Poly_2$ defined for a polynomial $f(x)$ by

$$T(f) = f'' + 2f$$

where f'' denotes the second derivative. Calculate $\det(T)$. Based on this, what can you say about $\ker(T)$.

Answer: Pick the standard basis $\{f_1 = 1, f_2 = x, f_3 = x^2\}$. Then the transformation has $T(f_1) = 2 = 2f_2$, $T(f_2) = 2x = 2f_2$, and $T(f_3) = 2 + 2x^2 = 2f_1 + 2f_3$. Thus the matrix expressing T in this basis is

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $\det(T) = \det(A) = 2 \times 2 \times 2 = 8$. Thus the map T is invertible and thus its kernel is trivial, i.e.

$$\ker(T) = \{0\}.$$

(c) (4 points) Find the determinant of the following matrix and say whether or not it is invertible.

$$\begin{pmatrix} 1 & 4 & 0 \\ 2 & 0 & 5 \\ 3 & 0 & 0 \end{pmatrix}.$$

Answer: Expanding along the bottom row gives that the determinant is

$$3 \times \begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 3 \times 4 \times 5 = 60.$$

Thus the matrix is indeed invertible.

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(d) (4 points) Suppose that A is a 2×2, and that $\text{Trace}(A) = 0$, while $\det(A) = -4$. Find the eigenvalues of A. (The trace $\text{Trace}(A)$ is the sum of the diagonal entries of A, but you may use any facts you know about it and the determinant.)

Answer: If the eigenvalues are λ_1, λ_2 then $\text{Trace}(A) = \lambda_1 + \lambda_2$. Since this is zero, $\lambda_1 = -\lambda_2$. On the other hand $\det(A) = \lambda_1\lambda_2$, so since this is -4 we have

$$-\lambda_1^2 = 4,$$

which means $\lambda_1 = 2$, so $\lambda_2 = -2$.

5. (20 points) **Eigenvalues and eigenvectors:**

   (a) (6 points) Complete the three equivalent definitions of diagonalizability. An $n \times n$ matrix A is diagonalizable if

   - there is a basis of $\mathbb{R}^n$ consisting of... **ans:** eigenvectors of A.

   - A is similar to a diagonal matrix, meaning... **ans:** there exists an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

   - All of the eigenvalues of A are real and for each one the algebraic multiplicity is ... **ans:** equal to the geometric multiplicity

   .

   (b) (2 points) Show that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is NOT diagonalizable. (You may just draw a picture)

   Answer: The eigenvalues are solutions to

$$\det\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda I_2\right) = \lambda^2 + 1 = 0,$$

   and are therefore not real numbers.'

(c) (12 points) Compute the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

For each eigenvalue, what are the algebraic and geometric multiplicities. Is the matrix diagonalizable?

Answer: First we compute the eigenvalues

$$0 = \det(M - \lambda I_3) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{pmatrix} = -\lambda(\lambda-2)^2.$$

So the eigenvalues are $\lambda = \{0, 2\}$, with

$$\text{alg mult}(0) = 1, \text{alg mult}(2) = 2.$$

To find the eigenspaces, we start with $\lambda = 0$ and compute

$$\ker(M - 0I_3) = \ker(M) = \ker(\text{Rref}(M)) = \ker \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the 0-th eigenspace is

$$\ker(M - 0I_3) = \text{span} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Next we do $\lambda = 2$.

$$\ker(M - 2I_3) = \ker \begin{pmatrix} -2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where in the last step we row reduced. Thus the $\lambda = 2$ eigenspace is

$$\ker(M - 2I_3) = \text{span} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The $\lambda = 2$ eigenspace is 2-dimensional, so

$$\text{geom mult}(0) = 1 = \text{alg mult}(0), \text{ while geom mult}(2) = 1 < \text{alg mult}(2).$$

Thus M is NOT diagonalizable.