LINEAR ALGEBRA (MATH 110.201)

MIDTERM II

Name: ____________________

Section number/TA: ____________________

Instructions:

(1) Do not open this packet until instructed to do so.
(2) This midterm should be completed in **50 minutes**.
(3) Notes, the textbook, and digital devices **are not permitted**.
(4) Discussion or collaboration is **not permitted**.
(5) All solutions must be written on the pages of this booklet.
(6) Justify your answers, and write clearly; points will be subtracted otherwise.

Exercise	Points	Your score
1	5	
2	5	
3	5	
4	5	
5	5	

Exercise 1 (5 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

(1) Find a basis for the subspace $\mathrm{Ker}(A) \subseteq \mathbb{R}^4$. What is $\dim(\mathrm{Ker}(A))$?
(2) Find a basis for the subspace $\mathrm{Im}(A) \subseteq \mathbb{R}^2$. What is $\dim(\mathrm{Im}(A))$?

Solution:

Exercise 2 (5 points) Let V be a vector space. Suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are all nonzero scalars. Show that if $v_1, \ldots, v_n \in V$ are linearly independent, then $\lambda_1 v_1, \ldots, \lambda_n v_n$ are also linearly independent. Will this conclusion still be true if we allow some of the $\lambda_1, \ldots, \lambda_n$ to be zero? Explain why or why not.

Solution:

Exercise 3 (5 points) Consider the following vectors as points in $\mathbb{R}^2$:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

The goal of this exercise is to find a quadratic polynomial g whose graph passes through these points (i.e., $g(1) = 1$, $g(2) = 2$, and $g(3) = -3$). Consider the following polynomials:

$$f_1(X) = \frac{(X-2)(X-3)}{(1-2)(1-3)} \quad f_2(X) = \frac{(X-1)(X-3)}{(2-1)(2-3)} \quad f_3(X) = \frac{(X-1)(X-2)}{(3-1)(3-2)}$$

(1) Show that f_1, f_2, f_3 are linearly independent in $P_2(\mathbb{R})$. Hint: Note that if $c_1 = 1$, $c_2 = 2$, and $c_3 = 3$, then $f_i(c_i) = 1$ and $f_i(c_j) = 0$ if $i \neq j$.
(2) Deduce that f_1, f_2, f_3 are a basis of $P_2(\mathbb{R})$.
(3) Use (2) to find a polynomial $g \in P_2(\mathbb{R})$ such that $g(1) = 1$, $g(2) = 2$, and $g(3) = -3$.

Solution:

Exercise 4 (5 points) Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix}$$

(1) What is the dimension of $W = \mathrm{Span}(v_1, v_2)$? (Explain carefully).
(2) Find an orthonormal basis of W.

Solution:

Exercise 5 (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation.

(1) Show that if $x \in \mathbb{R}^n$ and $x \neq \vec{0}_n$, then $T(x) \neq \vec{0}_n$.

(2) Show that if $x, y \in \mathbb{R}^n$, then the dot products $x \cdot y = T(x) \cdot T(y)$ are equal. Hint: You can use the fact that $T(e_1), \ldots, T(e_n)$ is an orthonormal basis of $\mathbb{R}^n$.

(3) Show, using (2), that if $x, y \in \mathbb{R}^n$ are nonzero vectors with angle θ between them, then the angle between $T(x)$ and $T(y)$ is equal to θ.

Solution: