THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences

110.201 - Linear Algebra
Midterm Exam - Spring Session 2010

Instructions: This exam has 13 pages. No calculators, books or notes allowed.

• **You must answer the first 3 questions, then answer one of question 4 or 5.** **Do not answer both. No extra points will be rewarded.**

Question 6 is bonus.

- **Place an "X" through the question you are not going to answer.**
- **You must use a pen.**

You have: 50 MINUTES.

Be sure to show all work for all problems. No credit will be given for answers without work shown. If you do not have enough room in the space provided you may use additional paper: **ask the Instructor to get additional paper**. If you use extra paper, be sure to clearly label each problem and attach the extra paper to the exam.

Academic Honesty Certification

I agree to complete this exam without unauthorized assistance from any person, materials or device.

Signature: ______________________________ Date: ____________________

Name of the student **AND session nb. (or TA's name)**: ____________________________

Problem	Score
1	
2	
3	
4 or 5	
6 (Bonus)	
Total	

1a. [10 points] Let $\mathcal{E} = \{\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$ be the canonical basis of $\mathbb{R}^3$. Let

$$\overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \overrightarrow{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \ \overrightarrow{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

- Prove that $\mathcal{B} = \{\overrightarrow{v}_1, \ \overrightarrow{v}_2, \overrightarrow{v}_3\}$ is a basis of $\mathbb{R}^3$.
- Write the matrix $S = S_{\mathcal{E} \to \mathcal{B}}$ that describes the change of basis from $\mathcal{E}$ to $\mathcal{B}$.

1b. [15 points] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\overrightarrow{e}_1) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ T(\overrightarrow{e}_2) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ T(\overrightarrow{e}_3) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

- Determine the matrix A of T (i.e. $T = T_A$) with respect to the canonical basis $\mathcal{E}$ and determine also the matrix B of T with respect to the basis $\mathcal{B}$ as in **1a.**, i.e. define explicitly $T_B : (\mathbb{R}^3, \mathcal{B}) \to (\mathbb{R}^3, \mathcal{B})$.
- Is T_B invertible?

2. [25 points] Given the subspace of $\mathbb{R}^{2\times 2}$ (= linear space of 2×2 real matrices)

$$S = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in \mathbb{R}^{2\times 2} \ : \ \begin{bmatrix} 1 & -\frac{4}{3} \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0\},$$

find its dimension and a basis $\mathcal{B}$ of S.

3. [25 points] Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix}$.

3a. Determine a basis of the column space of A.

3b. Determine a basis of the nullspace of A (i.e. a basis of $\mathrm{Ker}(A)$).

3c. For what value(s) of $r \in \mathbb{R}$ is the following system solvable

$$A\overrightarrow{x} = \begin{bmatrix} r \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

4. [25 points] (**ANSWER THIS QUESTION OR NUMBER 5**)

State whether the following statements are true or false. If true explain your answer, if false give an example for which the statement is false or motivate your answer:

(a) The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is similar to $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) The matrix $B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ defines a linear isomorphism $T_B : \mathbb{R}^3 \to \mathbb{R}^3$.

(c) If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear transformation, then $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is an orthonormal basis.

(d) If a matrix A is similar to an invertible matrix, then A is invertible.

5. [25 points] **(ANSWER THIS QUESTION OR NUMBER 4)**

State whether the following statements are true or false. If true explain your answer, if false give an example for which the statement is false:

(a) The matrix $M = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ has a QR factorization.

(b) If $\mathcal{A} = \{f, g\}$ and $\mathcal{B} = \{g, f - g\}$ are two bases of a linear space V, then the change of basis matrix from $\mathcal{B}$ to $\mathcal{A}$ is $S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

(c) If A and B are two 2×3 matrices with $\text{Image}(A) = \mathbb{R}^2 = \text{Image}(B)$, then $\text{Ker}(A) = \text{Ker}(B)$.

(d) If a matrix A is similar to a matrix B then $\text{Ker}(A)$ is isomorphic to $\text{Ker}(B)$.

6. [20 points] **(BONUS: ANSWER THIS QUESTION TO GET EXTRA POINTS)**

6a. Find an orthonormal basis for the subspace $V \subset \mathbb{R}^3$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

6b. Find a normal (i.e. unit) basis for the orthogonal complement of V.