Math 201
Spring 2014
Midterm 2
04/09/14
Lecturer: Jesus Martinez Garcia
Time Limit: 50 minutes

Name (Print): ____________________

Teaching Assistant ____________________

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a theorem of lemma you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) **Inner Product spaces and Gram-Schmidt.** Recall $C[-1,1]$ denotes all functions which are continuous on $[-1,1]$. Let $P_2 \subset C[-1,1]$, the space of polynomials of degree at most 2. Find an orthonormal basis for P_2 under the inner product given by:

$$\langle f, g\rangle = \frac{1}{2}\int_{-1}^{1} f(t)g(t)dt.$$

(Hint: you may apply Gram-Schmidt using inner product instead of dot product, as we did for Fourier approximations).

Solutions: $\{1, t, t^2\}$ is a basis of P_2. We apply Graham-Schmidt:

$$\langle 1, 1\rangle = \frac{1}{2}\int_{-1}^{1} 1dt = 1.$$

Therefore, let $u_1 = 1$.

$$\langle 1, t\rangle = \frac{1}{2}\int_{-1}^{1} tdt = \left.\frac{t^2}{4}\right|_{-1}^{1} = 0,$$

$$\langle t, t\rangle = \frac{1}{2}\int_{-1}^{1} t^2dt = \left.\frac{t^3}{6}\right|_{-1}^{1} = \frac{1}{3}.$$

Therefore, let $u_2 = \frac{t}{\|t\|} = \sqrt{3}t$. Observe that $\{u_1, u_2\}$ is a set of orthonormal vectors.

$$\langle 1, t^2\rangle = \frac{1}{2}\int_{-1}^{1} t^2dt = \left.\frac{t^3}{6}\right|_{-1}^{1} = \frac{1}{3},$$

$$\langle \sqrt{3}t, t^2\rangle = \frac{1}{2}\int_{-1}^{1} \sqrt{3}t^2dt = 0.$$

Therefore, let $u_3' = t^2 - \langle 1, t^2\rangle 1 - \langle \sqrt{3}t, t^2\rangle(\sqrt{3}t) = t^2 - \frac{1}{3}$. Observe that

$$\|u_3'\|^2 = \left\langle t^2 - \frac{1}{3}, t^2 - \frac{1}{3}\right\rangle = \frac{1}{2}\int_{-1}^{1}\left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right)dt = \left.\frac{t^5}{5} - \frac{2}{9}t^3 + \frac{1}{9}t\right|_{-1}^{1} = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$$

Hence, $\left\{1, \sqrt{3}t, \frac{\sqrt{45}}{2}\left(t^2 - \frac{1}{3}\right)\right\}$ is an orthonormal basis.

2. (25 points) **Linear transformations.** Recall that P is the space of polynomials in one variable. Let $T\colon P \to P$ be defined by $T(f(t)) = f''(t) + 2f'(t)$.

   (a) (10 points) Show that T is linear.

   Solution: By definition, we need to show that

$$T(f(t)+g(t)) = (f(t)+g(t))'' + 2\,(f(t)+g(t))' = \big(f''(t)+2f'(t)\big) + \big(g''(t)+2g'(t)\big)$$

$$= T(f(t)) + T(g(t)).$$

$$T(kf(t)) = \big(kf''(t)\big)'' + 2\,(kf(t))' = k\,\big(f''(t)+2f'(t)\big) = kT\,(f(t))\,.$$

   (b) (15 points) Find a basis for Ker(T).

   Solution: Let $f = a_0 + a_1x + \ldots + a_nx^n \in \text{Ker}(T)$. Then $T(f(t)) = 0$ if and only if

$$0 = (2a_2 + 2\cdot 3a_3x + \cdots + n\cdot(n-1)a_nx^{n-2} + 2\left(a_1 + 2a_2x + \cdots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1}\right) =$$

$$= (2a_2+2a_1) + (2\cdot 3a_3 + 2\cdot 2a_2)x + \cdots + (n\cdot(n-1)a_n + 2na_{n-1})x^{n-2} + 2na_nx^{n-1}.$$

   Hence $2a_2 + 2a_1 = 0, 2\cdot 3a_3 + 2\cdot 2a_2 = 0, \ldots, n\cdot(n-1)a_n + 2na_{n-1} = 0, 2na_n = 0$. Therefore $a_n = a_{n-1} = \cdots = a_2 = a_1 = 0$. So $f(t) = a_0$, $a_0 \in \mathbb{R}$. We conclude $\text{Ker}(T) = \{a_0 \mid a_0 \in \mathbb{R}\}$ and a basis of $\text{Ker}(T)$ is $\{1\}$.

3. (25 points) **The matrix of a linear transformation and dimension of vector spaces**

(a) (10 points) Find the matrix of the linear transformation $L\colon \mathrm{Mat}_2(\mathbb{R}) \to \mathrm{Mat}_2(\mathbb{R})$ given by $L(A) = A^t - 2A$ with respect to the basis

$$\mathfrak{B} = \left\{ v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ v_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Solution: Recall that the $\mathfrak{B}$-matrix of L is given by

$$B = \begin{pmatrix} | & | & | & | \\ [L(v_1)]_{\mathfrak{B}} & [L(v_2)]_{\mathfrak{B}} & [L(v_3)]_{\mathfrak{B}} & [L(v_4)]_{\mathfrak{B}} \\ | & | & | & | \end{pmatrix}$$

We compute all the $L(v_i)$ in terms of the elements of $\mathfrak{B}$:

$$L(v_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & -1 \end{pmatrix} = v_2 - 2v_1. \Rightarrow [L(v_1)]_{\mathfrak{B}} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$L(v_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix} = v_1 - 2v_2. \Rightarrow [L(v_2)]_{\mathfrak{B}} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

$$L(v_3) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = -v_3. \Rightarrow [L(v_3)]_{\mathfrak{B}} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

$$L(v_4) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} -2 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} = -v_4. \Rightarrow [L(v_4)]_{\mathfrak{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

We conclude

$$B = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(b) (15 points) Show that if W is a subspace of V and V is finite dimensional, then W must be finite dimensional as well.

Solution: Suppose $\dim(W) = \infty$. We want to achieve a contradiction. Let $n = \dim(V) < \infty$, $W \subseteq V$. Then there are $w_1, \ldots w_{n+1} \in W$ linearly independent vectors. But since $W \subseteq V$, then $w_1, \ldots w_{n+1}$ are linearly independent vectors in V, which is impossible since $\dim(V) = n$. This gives the desired contradiction.

4. (25 points) **Determinants and orthogonal matrices.**

   (a) (10 points) Compute

$$\det\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & -1 & 0 & 1\\ 0 & 0 & 1 & -1\\ 0 & -1 & 1 & 2\end{pmatrix}$$

Solution: By Laplace expansion:

$$\det\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & -1 & 0 & 1\\ 0 & 0 & 1 & -1\\ 0 & -1 & 1 & 2\end{pmatrix} = \det\begin{pmatrix}-1 & 0 & 1\\ 0 & 1 & -1\\ -1 & 1 & 2\end{pmatrix} = -\det\begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix} - \det\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix} = -(2+1)-(-1) = -2$$

   (b) (15 points) For each of the following statements decide if it is true or false. If it is true, prove it. If it is false, provide a counter-example.

   (i) If A is a symmetric square matrix of size n and S is an orthogonal matrix of size n, then $S^{-1}AS$ is symmetric.

   (ii) If A is an square matrix of size n such that $\|A\vec{u}\| = 1$ for all unit vectors $\vec{u}$, then A is orthogonal.

   (iii) The map $\det\colon \mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}$ is linear.

Solution:

   (i) True. Since S is orthogonal, then $S^{-1} = S^T$. A is symmetric if and only if $A^T = A$. Then:

$$(S^{-1}AS)^T = S^T A^T (S^{-1})^T = S^{-1} A (S^T)^T = S^{-1}AS.$$

   (ii) True: Let $\vec{v} \in \mathbb{R}^n$. Then

$$\|A\vec{v}\| = \left\| A\,\|\vec{v}\|\,\frac{\vec{v}}{\|\vec{v}\|}\right\| = \|\vec{v}\|,$$

   since $\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector.

   (iii) False. Let $n > 1$. Then $\det(2I) = 2^n \neq 2 = 2\det(I)$.