LINEAR ALGEBRA (MATH 110.201)

MIDTERM II - 1 APRIL 2016

Name: ______________________________

Section number/TA: ______________________________

Instructions:

(1) Do not open this packet until instructed to do so.
(2) This midterm should be completed in **50 minutes**.
(3) Notes, the textbook, and digital devices **are not permitted**.
(4) Discussion or collaboration is **not permitted**.
(5) All solutions must be written on the pages of this booklet.
(6) Justify your answers, and write clearly; **points will be subtracted otherwise**.
(7) Once you submit your exam, you will not be allowed to modify it.
(8) By submitting this exam, you are agreeing to the above terms. Cheating or refusing to stop writing when time is called may result in an automatic failure.

Exercise	Points	Your score
1	8	
2	12	
3	16	
4	12	
5	12	
6	12	
7	12	
8	16	
Total	100	

Exercise 1 (8 points). *Give an example of a* 2×2 *matrix* A *with* $\mathrm{Im}(A) \neq \mathrm{Im}(\mathrm{RREF}(A))$.

Solution:

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$,

$\mathrm{im}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ while $\mathrm{im}(\mathrm{rref}(A)) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

Obviously, $\mathrm{im}(A) \neq \mathrm{im}(\mathrm{rref}(A))$.

Exercise 2 (12 points). *Suppose that B is any $m \times n$ matrix having only $\vec{0}_n$ in its kernel, and suppose that C is any $n \times k$ matrix having only $\vec{0}_k$ in its kernel. Under these assumptions, can you describe all vectors in* $\mathrm{Ker}(BC)$*?*

Solution:

Set up the system $BC\vec{x} = \vec{0}_m \Rightarrow B(C\vec{x}) = \vec{0}_m$.

Since $\ker(B)$ consists of only $\vec{0}_n$, it follows that $C\vec{x} = \vec{0}_n$. Since $\ker(C)$ consists of only $\vec{0}_k$, we must then have $\vec{x} = \vec{0}_k$.

So $\ker(BC) = \{\vec{0}_k\}$.

Exercise 3 (16 points). *Consider the following matrix:*

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 4 & 2 & 3 & 4 \\ 3 & 4 & 5 & 3 & 4 & 5 \\ 4 & 5 & 6 & 4 & 5 & 6 \end{pmatrix}$$

(1) *(4 points) Compute* RREF(A).
(2) *(6 points) Give a basis of* Ker(A), *and write down* dim(Ker(A)).
(3) *(6 points) Give a basis of* Im(A), *and write down* dim(Im(A)).

Solution:

(1)

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 4 & 2 & 3 & 4 \\ 3 & 4 & 5 & 3 & 4 & 5 \\ 4 & 5 & 6 & 4 & 5 & 6 \end{bmatrix} \end{array} \begin{array}{l} \\ -2\mathrm{I} \\ -3\mathrm{I} \\ -4\mathrm{I} \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & -1 & -2 & 0 & -1 & -2 \\ 0 & -2 & -4 & 0 & -2 & -4 \\ 0 & -3 & -6 & 0 & -3 & -6 \end{bmatrix} \begin{array}{l} \\ \div(-1) \\ \\ \\ \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & -2 & -4 & 0 & -2 & -4 \\ 0 & -3 & -6 & 0 & -3 & -6 \end{bmatrix} \begin{array}{l} -2\mathrm{II} \\ \\ +2\mathrm{II} \\ +3\mathrm{II} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last matrix above is rref(A).

(2) The solution of the system $A\vec{x} = \vec{0}$ is

$$x_1 = x_3 - x_4 + x_6, \quad x_2 = -2x_3 - x_5 - 2x_6 \quad \text{or} \quad \begin{bmatrix} s - t + v \\ -2s - u - 2v \\ s \\ t \\ u \\ v \end{bmatrix} =$$

$$s\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

A basis of ker(A) is thus $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$;

dim ker(A) = 4.

(3) Pick the columns of A that correspond to columns of rref(A) containing leading 1's:

A basis of im(A) is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$; dim im($A$) = 2.

Exercise 4 (12 points). *If A is a nonzero 2×5 matrix, can we have* $\dim(\mathrm{Ker}(A)) = 1$*? What are the possibilities for* $\dim(\mathrm{Ker}(A))$*?*

Solution:

No, we can't have $\dim \ker(A) = 1$.

By the rank-nullity theorem, $\mathrm{rank}(A) + \dim \ker(A) = 5$.

Since A has 2 rows, $\mathrm{rank}(A) \leqslant 2$. It then follows that $\dim \ker(A) \geqslant 3$. So we can't have $\dim \ker(A) = 1$.

By the above analysis, the possibilities for $\dim \ker(A)$ are:

$\dim \ker(A) = 3 \quad (\Leftrightarrow \mathrm{rank}(A) = \dim \mathrm{im}(A) = 2)$

e.g. $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$;

$\dim \ker(A) = 4 \quad (\Leftrightarrow \mathrm{rank}(A) = \dim \mathrm{im}(A) = 1)$

e.g. $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$;

$\dim \ker(A) = 5 \quad (\Leftrightarrow \mathrm{rank}(A) = \dim \mathrm{im}(A) = 0)$

$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Exercise 5 (12 points). *Determine whether the following vectors are a basis of* $\mathbb{R}^4$ *(justify your answer):*

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\2\\2\end{bmatrix}, \quad \begin{bmatrix}1\\2\\3\\3\end{bmatrix}, \quad \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

Solution:

Since $\dim \mathbb{R}^4 = 4$ and we have 4 vectors in $\mathbb{R}^4$, to determine whether the vectors are a basis of $\mathbb{R}^4$, it suffices to determine whether they are linearly independent. So set up the system

$$c_1 \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_2 \begin{bmatrix}1\\2\\2\\2\end{bmatrix} + c_3 \begin{bmatrix}1\\2\\3\\3\end{bmatrix} + c_4 \begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

and solve it:

$$\begin{bmatrix}1&1&1&1\\1&2&2&2\\1&2&3&3\\1&2&3&4\end{bmatrix}\begin{array}{l}\\-\mathrm{I}\\-\mathrm{I}\\-\mathrm{I}\end{array} \to \begin{bmatrix}1&1&1&1\\0&1&1&1\\0&1&2&2\\0&1&2&3\end{bmatrix}\begin{array}{l}-\mathrm{II}\\ \\-\mathrm{II}\\-\mathrm{II}\end{array} \to$$

$$\begin{bmatrix}1&0&0&0\\0&1&1&1\\0&0&1&1\\0&0&1&2\end{bmatrix}\begin{array}{l}\\-\mathrm{III}\\ \\-\mathrm{III}\end{array} \to \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{array}{l}\\ \\-\mathrm{IV}\\ \\ \end{array} \to \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

The above system has a unique solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$.

Therefore, the vectors are linearly independent, and hence form a basis of $\mathbb{R}^4$.

Exercise 6 (12 points). *Let $\mathscr{B}$ denote the basis $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ of $\mathbb{R}^2$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation whose matrix with respect to $\mathscr{B}$ is $\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$. What is the matrix of T with respect to the standard basis of $\mathbb{R}^2$? Express your answer in terms of a, b, c, d.*

Solution:

That the matrix of T with respect to $\mathcal{B}$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ means

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = a\begin{bmatrix}1\\0\end{bmatrix} + c\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a+c\\c\end{bmatrix} \quad \text{and}$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = b\begin{bmatrix}1\\0\end{bmatrix} + d\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}b+d\\d\end{bmatrix}.$$

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} \Rightarrow T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$= \begin{bmatrix}b+d-a-c\\d-c\end{bmatrix}.$$

The two columns of the matrix of T with respect to the standard basis are

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}a+c\\c\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}b+d-a-c\\d-c\end{bmatrix};$$

that matrix is thus

$$\begin{bmatrix}a+c & b+d-a-c\\ c & d-c\end{bmatrix}.$$

- Let $S = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ be the matrix whose two columns are the two vectors in $\mathcal{B}$, i.e. the change of basis matrix from $\mathcal{B}$ to the standard basis $\mathcal{A}$. Then the $\mathcal{B}$-matrix B of T and the $\mathcal{A}$-matrix A of T are related by $A = SBS^{-1}$. You might also apply this formula to find A.

Exercise 7 (12 points). *Suppose that v_1, v_2, v_3, v_4 are linearly independent vectors in a linear space V. Show that the vectors*

$$v_1, \quad v_2 + v_1, \quad v_3 + v_2 + v_1, \quad v_4 + v_3 + v_2 + v_1$$

are also linearly independent in V.

Solution:

Set up the equation

$$c_1 v_1 + c_2 (v_2 + v_1) + c_3 (v_3 + v_2 + v_1) + c_4 (v_4 + v_3 + v_2 + v_1) = 0$$

$$\Rightarrow \quad (c_1 + c_2 + c_3 + c_4) v_1 + (c_2 + c_3 + c_4) v_2 + (c_3 + c_4) v_3 + c_4 v_4 = 0$$

where c_1, c_2, c_3, c_4 are unknown scalars.

Since v_1, v_2, v_3, v_4 are linearly independent, it follows that

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 0 \\ c_2 + c_3 + c_4 &= 0 \\ c_3 + c_4 &= 0 \\ c_4 &= 0. \end{aligned}$$

Solving this system from the last equation all the way to the first, we obtain successively

$$c_4 = 0, \quad c_3 = 0, \quad c_2 = 0, \quad c_1 = 0.$$

Therefore, v_1, $v_2 + v_1$, $v_3 + v_2 + v_1$, $v_4 + v_3 + v_2 + v_1$ are linearly independent.

Exercise 8 (16 points). *Consider the set $W \subseteq P_3(\mathbb{R})$ of polynomials $f(X)$ of degree ≤ 3 having the property that $f(-2) = 0$.*

(1) *(4 points) Give an example of a degree 3 polynomial in W. Give an example of a degree 3 polynomial which is not in W.*
(2) *(6 points) Show that W is a subspace of $P_3(\mathbb{R})$.*
(3) *(6 points) Find a basis of W.*

Solution:

(1) $f(x) = x^3 + 8$ is in W and of degree 3;

$g(x) = x^3$ is a degree 3 polynomial not in W.

(2) For any $f(x)$, $g(x)$ in W and any scalar k,

$f(x) + g(x)$ has degree ≤ 3 and

$f(-2) + g(-2) = 0$ $\Rightarrow$ $f(x) + g(x)$ is in W;

$kf(x)$ has degree ≤ 3 and

$kf(-2) = 0$ $\Rightarrow$ $kf(x)$ is in W.

Finally, the zero polynomial is in W (don't forget this!!).

Therefore, W is a subspace of $P_3(\mathbb{R})$.

(3) Suppose $f(x) = a + bx + cx^2 + dx^3$ is in W. Then

$f(-2) = 0 \Rightarrow a - 2b + 4c - 8d = 0 \Rightarrow a = 2b - 4c + 8d$.

Thus a typical element in W can be written as

$2b - 4c + 8d + bx + cx^2 + dx^3$

$= b(2 + x) + c(-4 + x^2) + d(8 + x^3)$

where b, c, d are arbitrary.

Obviously, $2 + x$, $-4 + x^2$, $8 + x^3$ are linearly independent,

so a basis of W is

$2 + x$, $-4 + x^2$, $8 + x^3$.