Midterm Exam 2 — Apr. 14, 2017

1. (a) (15 points) Find a matrix S that shows that

$$A = \begin{bmatrix} -49 & 80 \\ -30 & 49 \end{bmatrix} \text{ is similiar to } D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) (5 points) Compute A^{10}.

2. Determine which of the following transformations with domain P_2, the space of all polynomials of degree at most 2, is a linear isomorphism. Remember to justify your answers.

   (a) (5 points) $T_1 : P_2 \to \mathbb{R}^4$ defined by $T_1(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{bmatrix}$.

   (b) (5 points) $T_2 : P_2 \to \mathbb{R}^3$ defined by $T_3(p) = \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \end{bmatrix}$.

(c) (5 points) $T_3 : P_2 \to \mathbb{R}^3$ defined by $T_2(p) = \begin{bmatrix} p(1)+2 \\ (p(0))^3 \\ p'(0) \end{bmatrix}$.

(d) (5 points) $T_4 : P_2 \to P_2$ defined by $T_4(p)(x) = xp'(x)$.

3. (a) (10 points) Let $\mathbb{R}^{2\times 2}$ be the space of 2×2 matrices and consider the ordered basis, $\mathcal{B}$, of $\mathbb{R}^{2\times 2}$,

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

For the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by

$$T(A) = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} A - A \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix},$$

determine $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix of T.

(b) (10 points) Find a basis of $\operatorname{im}(T) \subset \mathbb{R}^{2\times 2}$.

4. (a) (10 points) Determine *all* $a_1, a_2, a_3 \in \mathbb{R}$ so that the following is an orthogonal matrix:

$$Q = \frac{1}{7}\begin{bmatrix} a_1 & 2 & 6 \\ a_2 & -6 & 3 \\ a_3 & 3 & 2 \end{bmatrix}.$$

(b) (10 points) Suppose a matrix, M, has the QR-factorization $M = QR$. Determine R given

$$M = \begin{bmatrix} 4 & -1 & 0 \\ 4 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} \text{ and } Q = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ 1 & -2 & -2 \end{bmatrix}.$$

5. In what follows, determine if the matrix C is symmetric, skew-symmetric or if there is not enough information to decide. Remember to justify your answer

   (a) (5 points) $C = QAQ^{-1}$ where $A \in \mathbb{R}^{n\times n}$ is symmetric and $Q \in \mathbb{R}^{n\times n}$ is orthogonal.

   (b) (5 points) $C = ABA$, where $A, B \in \mathbb{R}^{n\times n}$ are both skew-symmetric.

(c) (5 points) $C = A^\top A - AA^\top$, where $A \in \mathbb{R}^{n\times n}$

(d) (5 points) $C = I_n + P^2$ where $P \in \mathbb{R}^{n\times n}$.