Lecture 1: Overview+Review of vector operations.

**What is the multivariable calculus course about?**
Curves in space \((x(t), y(t), z(t))\), e.g. path of a particle.
Vectors and vector operations, e.g. the dot and the cross product.
Equations of lines and planes. (using vectors)
The the velocity \(v(t) = (x'(t), y'(t), z'(t))\), tangent vector to the particle.
Functions of several variables \(f(x, y, z)\), e.g. temperature at each point in space.
Derivatives of functions of several variables:
E.g. if \(f\) represent the temperature then the gradient vector \(\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\)
represents the direction in which the temperature increases the most.
Multiple integrals: Double integrals over domains in the plane and triple or volume integrals over domains in space. Polar coordinates.

Ch 1: Vectors and vector operations.
Ch 2: Derivatives of functions of several variables.
Ch 3: **Vector fields** \(v(x, y, z) = ((v_1(x, y, z), v_2(x, y, z), v_3(x, y, z))\), at each point in space we are given a vector, and the divergence and curl of them.
The gravitational field is a vector field; at each point in space we are given a vector.
The vector field of a continuum of fluid particles is a vector field. At each point in space we are given the velocity of the particles at that point.

Each component of a vector field can be differentiated in each direction.
The divergence and curl are special derivatives with physical meaning.
Ch 4: Mx/min in several variables.
Ch 5 Double and Triple Integrals and the **change of variable theorem** in multiple integrals.
Ch 6 Line Integrals and Green’s theorem. Generalization of the fundamental theorem of calculus \(\int_a^b f'(x) \, dx = f(b) - f(a)\) to several variables: E.g. \(F = (F_1, F_2, 0)\)
a vector function and \(D\) a domain in the \(x-y\) plane with boundary \(C\) then
\[
\int_C F_1 \, dx + F_2 \, dy = \iint_D (\partial F_2/\partial x - \partial F_1/\partial y) \, dx.
\]
Ch 7 **Surface area and surface Integrals** flow through surface and Stoke’s and Gauss’s theorems.
Ch 8: **Differential forms.**

Surface integrals is the hardest part. Thinking geometrically and physically helps understanding concepts but is not needed to do problems.
Section 1.1 Vectors.

Recall that a point in space can be represented by an ordered triplet $(x, y, z)$ of real numbers called the Cartesian or rectangular coordinates, that measure the lengths of the projections on the three coordinate axis.

A vector $\mathbf{a}$ is a directed line segment or arrow; it has a length $\|\mathbf{a}\|$ and a direction. Its components $(a_1, a_2, a_3)$ are the coordinates of the endpoint if it starts at the origin. In the notes vectors will be denoted by boldface letters $\mathbf{a}$ and in the lectures by $\overrightarrow{a}$.

Addition of vectors is geometrically defined by the triangle law and algebraically by $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Scalar multiplication $\alpha \mathbf{a}$ is a vector in the same direction as $\mathbf{a}$ with length $|\alpha| \|\mathbf{a}\|$ and its algebraically given by $\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$.

The length of the vector is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, by the Pythagorean theorem. A unit vector (i.e. of length one) in the direction of the vector $\mathbf{a}$ is given by $\frac{\mathbf{a}}{\|\mathbf{a}\|}$.

Section 1.2: Basis and what we can do with vectors?

Standard basis: If $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ then $\mathbf{a} = (a_1, a_2, a_3) = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$.

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are points in space then the vector represented by the directed line segment $\overrightarrow{P_1 P_2}$ is $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

Ex Find the parametric equations of a line passing through $(0, 1, 2)$ and $(1, 2, 4)$.

Sol The vector $\mathbf{v} = (1, 2, 4) - (0, 1, 2) = (1, 1, 2)$ is parallel to the line and so is any multiple $t\mathbf{v}$. The points on the line are therefore given by $(x, y, z) = (0, 1, 2) + t(1, 1, 2) = (t, 1 + t, 2 + 2t)$.

Section 1.3:. Inner product (or scalar or dot product) of two vectors is

$$a \cdot b = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The geometric interpretation is $\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Note that $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$ and that $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Note that $\|\mathbf{b}\| \cos \theta$ is the component of $\mathbf{b}$ in the direction of $\mathbf{a}$.

The orthogonal projection $\mathbf{p}$ of $\mathbf{b}$ on $\mathbf{a}$ is given by $\mathbf{p} = \|\mathbf{b}\| \cos \theta \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a}$.

Ex. Decompose $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ into a vector $\mathbf{b}_\parallel$ parallel to $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and a vector $\mathbf{b}_\perp$ perpendicular to $\mathbf{a}$.

Sol. $\mathbf{a} \cdot \mathbf{b} = 2 - 2 + 4 = 4$, $\|\mathbf{a}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ so $\mathbf{b}_\parallel = (\mathbf{a} \cdot \mathbf{b}) \frac{\mathbf{a}}{\|\mathbf{a}\|^2} = 4(\mathbf{i} + \mathbf{j} + \mathbf{k})/3$ and $\mathbf{b}_\perp = \mathbf{b} - \mathbf{b}_\parallel = (2 - 4/3)\mathbf{i} - (1 + 4/3)\mathbf{j} + (4 - 4/3)\mathbf{k}$.
Section 1.4. The vector or cross product is the vector

\[ \mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{i} + (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}. \]

The geometric interpretation is \( \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta \mathbf{n} \), where \( \mathbf{n} \) is a unit vector, \( \|\mathbf{n}\| = 1 \), that is perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \) and pointing in the direction so that \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{n} \) form a positively oriented system.

Note that \( \mathbf{a} \times \mathbf{b} = 0 \) if and only if \( \mathbf{a} \) and \( \mathbf{b} \) are parallel.

To remember the definition of vector product we introduce so called determinants. A determinant of order 2 is defined by

\[
\begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix} = ad - bc
\]

Its magnitude is the area of the parallelogram with vectors \((a, b)\) and \((c, d)\) as edges. A determinant of order 3 is defined by

\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3
\end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}
\]

Its magnitude is the volume of the parallelepiped with vectors \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \), \( \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \) and \( \mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \) as edges.

The cross product (2) is

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}
\]

Because of the similarity with (3), to remember this we symbolically write

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\]