Lecture 11: 4.3 Lagrange multipliers.

From previous lecture: In particular in one dimension. If we want to find the maximum of $f(x)$ over the interval $I = [a, b] = \{x;\ a \le x \le b\}$, then we first find all the critical points $f'(c_i) = 0$, $i = 1, ..., N$ and we check the value of f on these points and the boundary points a and b in order to find the largest and smallest value.

Ex. Find the max and min of $f(x, y) = x^2 + 2y^2$ over
(a) $D = \{(x, y);\ |x| \le 1,\ |y| \le 1\}$ and (b) $D = \{(x, y);\ x^2 + y^2 \le 1\}$.

Sol. Critical points $f_x = 2x = 0$ and $f_y = 4y = 0$ is $(x, y) = (0, 0)$ and $f(0, 0) = 0$.
(a) Extreme value on the boundary. Divide boundary into the 4 parts. (1) $x=1$ and $-1 \le y \le 1$: If $g(y)=f(1, y)=1+2y^2$ then $g'(y)=4y=0$ if $y=0$ and $g(0)=f(1, 0)=1$. Endpoints of the interval: $g(1) = f(1, 1) = f(1, -1) = g(-1) = 3$. (2) $y=1$ and $-1 \le x \le 1$. If $h(x)=f(x, 1)=x^2+1$ then $h'(x)=2x=0$ if $x = 0$ and $g(0)=f(0, 1)=2$. Endpoints of the interval: $h(1)=f(1, 1)=f(-1, 1)=h(-1)=3$. The other two parts of the boundary are the same so max is $f(\pm1, \pm1)=3$ and min is $f(0, 0)=0$.
(b) Extreme value on the boundary. Sol. 1: Parameterize boundary $(x, y) = (\cos t, \sin t)$. $g(t)=f(\cos t, \sin t)=\cos^2 t + 2\sin^2 t$. $g'(t)=-2\cos t \sin t + 4\sin t \cos t = 2\sin t \cos t = 0$ if $t = 0, \pi/2, \pi, 3\pi/2$. $g(0) = f(1, 0) = 1$, $g(\pi/2) = f(0, 1) = 2$, $g(\pi) = f(-1, 0) = 1$ and $g(3\pi/2) = f(0, -1) = 2$ so max is $f(0, \pm1) = 2$ and min is $f(0, 0) = 0$. Sol2: Solve for y in $x^2 + y^2 = 1$ gives $y = \pm\sqrt{1 - x^2}$, $-1 \le x \le 1$. Substituting into $f(x, y)$ gives $h(x) = f(x, \pm\sqrt{1 - x^2}) = 2 - x^2$ and we want to maximize over $-1 \le x \le 1$. $h'(x) = -2x = 0$ if $x = 0$ and $h(0) = f(0, \pm1) = 2$. Endpoints $h(\pm1)=f(\pm1, 0)=1$.

In the previous section we found the extreme of $f(x, y) = x^2 + 2y^2$ subject to the constraint that (x, y) was on the circle $x^2 + y^2 = 1$. We solved this problem by reducing it to a problem in one dimension less by using the constraint to solve for one of the variables in terms of the other. However, there is another general geometric method called Lagrange method.

Say that we want to maximize $f(x, y)$ subject to the constraint that $g(x, y) = 0$. Let C be the curve $g(x, y) = 0$. To maximize f on C is to find the level curve $f(x, y) = k$ with largest k that intersects C. Say that k_0 is the largest value and let C_0 by the level curve $f(x, y) = k_0$. It is geometrically clear that such a level curve C_0 has to be tangential to the curve C at the point $P_0(x_0, y_0)$ with maximum value $f(x_0, y_0) = k_0$ where they intersect. In fact, assume that they are not tangential, then since $f(x, y) > k_0$ on one side of C_0 it follows that C can not go in that region. Since at the point P_0 the level curves C and C_0 are tangential it follows that the gradients ∇f and ∇g have to be parallel as well:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{1}$$

Another way to see this is to parameterize the curve C: $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Then $h(t) = f(x(t), y(t))$ has a maximum at $t = t_0$ where $\mathbf{r}(t_0) = (x_0, y_0)$. Hence $h'(t_0) = \nabla f \cdot \mathbf{r}'(t_0) = 0$, i.e. if the gradient is orthogonal to the tangent line of C, but we already know that the gradient ∇g is orthogonal to C from section 14.6. Also the constraint has to be satisfied

$$g(x_0, y_0) = 0 \tag{2}$$

Lagrange method is to find the solution of (1) and (2).
Each point satisfying these two equations has to to be a local extreme value.

Ex. Find the max of $f(x,y)=x^2+2y^2$ subject to the constraint $g(x,y)=x^2+y^2=1$.
Sol. (1) become $\langle 2x,4y\rangle=\lambda\langle 2x,2y\rangle$ or $2x=\lambda 2x$ and $4y=\lambda 2y$. First we must have $\lambda\neq 0$ since $(0,0)$ is not on the boundary. Then if $x\neq 0$ it follows that $\lambda=1$ and hence $y=0$ and if $y\neq 0$ it follows that $\lambda=1/2$ and $x=0$ so the only possibilities are $(x,0)$ or $(0,y)$ Putting this into the constraint gives $(\pm 1,0)$ or $(0,\pm 1)$.

Recall that to find the max/min we first find the critical points $f_x(a,b) = f_y(a,b) = 0$ and then determine if they are max/min by using e.g. the second derivative test.

Ex. Find the minimum distance from the point $(1,2,2)$ to the plane $x+y+z=4$.

Sol. We want to minimize $F(x,y,z) = d(x,y,z)^2 = (x-1)^2+(y-2)^2+(z-2)^2$ over all (x,y,z) on the plane. Since on the plane $z = 4-x-y$ we can substitute for z and instead minimize $f(x,y) = F(x,y,4-x-y) = (x-1)^2+(y-2)^2+(x+y-2)^2$. Then $f_x = 2(x-1)+2(x+y-2) = 0$ and $f_y = 2(y-2)+2(x+y-2) = 0$ is equivalent to $4x+2y-6=0$ and $2x+4y-8=0$ which gives $(x,y) = (2/3,5/3)$ and $f(2/3,5/3) = 1/3$. By the second derivative test: $f_{xx}f_{yy}-f_{xy}^2 = 4\cdot 4-2^2 = 12 > 0$ and $f_{xx} > 0$ so it is a local min. That this indeed is the absolute minimum follows since there has to be a point on the plane with smallest distance.

Lagrange multipliers. To find the max and min of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ we find all values of (x,y,z) and λ such that

$$\nabla f(x,y,z) = \lambda\nabla g(x,y,z), \qquad \text{and} \qquad g(x,y,z) = k \tag{3}$$

Evaluating $f(x,y,z)$ at all the resulting points gives the max and min.

That (3) has to hold on a max can be argued as follows. Suppose that $f(x_0,y_0,z_0) = k_0$ is the maximum value and let $S_0 = \{(x,y,z);\ f(x,y,z) = k_0\}$ and $S = \{(x,y,z);\ g(x,y,z) = k\}$. Since the gradient is normal to the tangent planes the statement (3) is equivalent to saying that the tangent planes to the two surfaces has to be parallel at (x_0,y_0,z_0). However, this is geometrically clear since the surface S has to lie completely within the region $D_0=\{(x,y,z);\ f(x,y,z)\leq k_0\}$

Ex. Find the min of $F(x,y,z) = d(x,y,z)^2 = (x-1)^2+(y-2)^2+(z-2)^2$ subject to the constraint $g(x,y,z) = x+y+z = 4$.

Sol. $2(x-1) = \lambda 1$, $2(y-2) = \lambda 1$, $2(z-2) = \lambda 1$ so $x = 1+\lambda/2$, $y = 2+\lambda/2$, o $z = 2+\lambda/2$ Plug into the constraint gives $g(1+\lambda/2,2+\lambda/2,2+\lambda/2) = 5+3\lambda/2 = 4$ so $\lambda = -2/3$, which gives $(x,y,z) = (2/3,5/3,5/3)$ and $F(2/3,5/3,5/3) = 1/3$.

To find the max and min of $f(x,y,z)$ subject to the constraints $g(x,y,z) = k$ and $h(x,y,z) = c$ we find all values of (x,y,z), λ and μ such that

$$\nabla f(x,y,z) = \lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z), \quad \text{and} \quad g(x,y,z) = k, \quad h(x,y,z) = c \tag{4}$$

Evaluating $f(x,y,z)$ at all the resulting points gives the max and min.
Ex. Find the maximum volume of a rectangular box with side lengths x, y and z whose total area is 1500 cm^2 and the total length of the edges is 200 cm.
Sol. Maximize $V(x,y,z) = xyz$ subject to the constraint $A(x,y,z) = 2(xy+yz+zx) = 1500$ and $L(x,y,z) = 4(x+y+z) = 200$, Find (x,y,z), λ and μ such that

$$\nabla V(x,y,z)=\lambda\nabla A(x,y,z)+\mu\nabla L(x,y,z), \quad A(x,y,z)=1500, \quad \text{and} \quad L(x,y,z)=200$$