Lecture 17: 7.1 Parameterized surface. A Parameterized surface is given in terms of two parameters

\[ x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad \text{or} \quad \mathbf{T}(u, v) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \]

A particular example of a parameterized surface is a graph:

\[ z = f(x, y), \quad \text{or} \quad \mathbf{T}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)k \]

Ex The sphere \( x^2 + y^2 + z^2 = r^2 \) can be parameterized using spherical coordinates:

\[ x = r\sin \phi \cos \theta, \quad y = r\sin \phi \sin \theta, \quad z = r\cos \phi, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \pi \]

It can however, not be written as one graph, but one for the southern hemisphere \( z = -\sqrt{r^2 - x^2 - y^2} \) and one for the northern hemisphere \( z = \sqrt{r^2 - x^2 - y^2} \).

A surface is locally close to its tangent plane which is determined by its normal that we now will find. Another description of a surface is a level surface \( h(x, y, z) = 0 \), if \( \nabla h(x, y, z) \neq 0 \).

(The graph is a special case with \( h(x, y, z) = z - f(x, y) \).) In this case a normal is \( \mathbf{N} = \nabla h \)

In particular the usual way to describe a plane is as a level surface:

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \]

where \( \mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \) is a normal to the plane and \( \mathbf{c}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} \) is a point in the plane. However we can also give the parametric equations of a plane

\[ \mathbf{T}(u, v) = A\mathbf{u} + B\mathbf{v} + \mathbf{c}_0, \]

where \( \mathbf{A} \) and \( \mathbf{B} \) are vectors in the plane. To go from the parametric equations to the usual equation we note that the normal to the plane is given by \( \mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v \).

To find the unit normal to a parameterized surface recall that for a parameterized curve we found the tangent by differentiating with respect to the parameter. Here, \( u \to \mathbf{T}(u, v_j) \), where \( v_j \) is kept constant and \( u \) vary, is a parameterized curve and

\[ \mathbf{T}_u = \frac{\partial \mathbf{T}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \]

is tangent to this curve, and hence to the surface. Similarly the vector

\[ \mathbf{T}_v = \frac{\partial \mathbf{T}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} \]

is tangent to the curves \( v \to \mathbf{T}(u_i, v) \), and hence to the surface. The tangent plane to the surface is spanned by \( \mathbf{T}_u \) and \( \mathbf{T}_v \) so a normal to the surface is given by

\[ \mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v \]

Ex 2 Find a normal to the surface \( x = u\cos v, \ y = u\sin v, \ z = u \).

Sol \( \mathbf{T} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u\mathbf{k} \) so \( \mathbf{T}_u = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k}, \mathbf{T}_v = -u\sin v\mathbf{i} + u\cos v\mathbf{j} \).

\[ \mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = \cdots = -u\cos v\mathbf{i} - u\sin v\mathbf{j} + u\mathbf{k}. \]
Area of a parameterized surface.

A **Parameterized surface** is given in terms of two parameters

\[ x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad \text{or} \quad T(u, v) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \]

Now \( u \to T(u, v_j) \) and \( v \to T(u_i, v) \) are curves on the surface and hence

\[
T_u = \frac{\partial T}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \text{and} \quad T_v = \frac{\partial T}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.
\]

are **tangent vectors** to the surface. For \((u, v) \sim (u_i, v_j)\) the surface is close to its tangent plane at the point \((u_i, v_j)\) and \(T\) is close to the **linear approximation**:

\[ T(u, v) \sim L_{ij}(u, v) = T(u_i, v_j) + T_u(u_i, v_j)(u - u_i) + T_v(u_i, v_j)(v - v_j) \]

The image \(S_{ij} = T(R^*_{ij})\) of a small rectangle

\[ R^*_{ij} = \{(u, v); u_i \leq u \leq u_i + \Delta u, v_j \leq v \leq v_j + \Delta v\}. \]

under the map \(T\) is close to the image under the linear map \(L_{ij}\). The image of \(R^*_{ij}\) under \(L_{ij}\) is a parallelogram with adjacent sides \(T_u\Delta u\) and \(T_v\Delta v\) so

\[ \text{Area } (S_{ij}) \sim \|T_u \times T_v\| \Delta u \Delta v = \|T_u \times T_v\| \text{Area } (R^*_{ij}) \]

Summing up over all small rectangles in the \(u-v\) plane we get

\[ \text{Area } (S) = \sum \text{Area } (S_{ij}) \sim \sum \|T_u \times T_v(u_i, v_j)\| \Delta u \Delta v \]

and in the limit as \(\Delta u, \Delta v \to 0\) we get the formula for the **surface area** of a parameterized surface:

\[ \text{Area } (S) = \int \int \|T_u \times T_v(u, v)\| \, du \, dv \]

**Ex.** Find the area of the sphere \(S\) of radius \(r\).

**Sol.** Using the parametrization \(T = r\sin \phi \cos \theta \mathbf{i} + r\sin \phi \sin \theta \mathbf{j} + r\cos \phi \mathbf{k}\) we get

\[ T_\phi = r\cos \phi \cos \theta \mathbf{i} + r\cos \phi \sin \theta \mathbf{j} - r\sin \phi \mathbf{k} \quad \text{and} \quad T_\theta = -r\sin \phi \sin \theta \mathbf{i} + r\sin \phi \cos \theta \mathbf{j}; \]

\[ T_\phi \times T_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos \phi \cos \theta & r\cos \phi \sin \theta & -r\sin \phi \\ -r\sin \phi \sin \theta & r\sin \phi \cos \theta & 0 \end{vmatrix} = r^2 \sin^2 \phi \cos \theta \mathbf{i} + r^2 \sin^2 \phi \sin \theta \mathbf{j} + r^2 \sin \phi \cos \phi \mathbf{k} \]

and \(|T_\phi \times T_\theta| = r^2 \sin \phi \sqrt{\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \cos^2 \phi} = r^2 \sin \phi\). Hence

\[ \text{Area } (S) = \int_0^{2\pi} \int_0^\pi r^2 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi -r^2 \cos \phi \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi 2r^2 \, d\theta = 4\pi r^2. \]
**Surface area of a graph.** In the special case of a graph \( z = f(x, y) \) we have \( T(x, y) = x \mathbf{i} + y \mathbf{j} + f(x, y) \mathbf{k} \) we have \( T_x = \mathbf{i} + f_x(x, y) \mathbf{k}, \ T_y = \mathbf{j} + f_y(x, y) \mathbf{k} \) and

\[
T_x \times T_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}
\]

and

\[
\|T_x \times T_y\| = \sqrt{1 + f_x^2 + f_y^2}
\]

and hence we get the formula for the area of a graph:

\[
\text{Area (S)} = \int \int \sqrt{1 + f_x^2 + f_y^2} \ dx \ dy
\]

There is however a simpler way to remember this formula. Let

\[
R_{ij}^* = \{(x, y); x_i \leq x \leq x_i + \Delta x, y_j \leq y \leq y_j + \Delta y\},
\]

be a small rectangle in the \( x-y \) plane. Above this rectangle is a parallelogram in the tangent plane to the surface at \((x_i, y_j, f(x_i, y_j))\), that projects down to the rectangle in the \( x-y \) plane. The quotient of the area of the rectangle in the \( x-y \) plane to the area of the parallelogram in the tangent plane above it is the cosine of the angle \( \gamma \) between the tangent plane and the \( x-y \) plane. Hence

\[
\text{Area (S)} = \int \int \frac{dxdy}{|\cos \gamma|}
\]

If \( \mathbf{n} \) is the unit normal to the tangent plane and \( \mathbf{k} \) is the normal to the \( x-y \) plane then the angle is given by \( \cos \gamma = \mathbf{n} \cdot \mathbf{k} \). The unit normal to a graph \( z = f(x, y) \) is easiest calculated by writing it in the form \( h(x, y, z) = z - f(x, y) = 0 \):

\[
\mathbf{n} = \frac{\nabla h}{\|\nabla h\|} = \frac{-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}
\]

If we also take the inner product with \( \mathbf{k} \) the desired formula follows since

\[
|\cos \gamma| = |\mathbf{n} \cdot \mathbf{k}| = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}
\]

**Ex** Find the area of the part of the cone \( S = \{(x, y, z); z = \sqrt{x^2 + y^2}, x^2 + y^2 \leq 1\} \).

**Sol** The surface is a graph and the angle between the surface and the \( x-y \) plane is \( \gamma = \pi/4 \), since when say \( y = 0 \) its just \( z = |x| \). Hence \( \cos \gamma = 1/\sqrt{2} \). Alternatively one can calculate \( |\mathbf{n} \cdot \mathbf{k}| \). Hence

\[
\text{Area(S)} = \int_{x^2+y^2 \leq 1} \frac{dxdy}{|\cos \gamma|} = \sqrt{2} \int_{x^2+y^2 \leq 1} dxdy = \sqrt{2} \times \text{Area of unit disc} = \sqrt{2}\pi.
\]