Lecture 19: 7.2 Derivation of the flow integral. Let \( S \) be a closed surface and for each point \((x,y,z)\) on the surface let \( f(x,y,z) \) be the rate of flow of fluid out through the surface per unit surface area and unit time, i.e. the flow out of a small area \( \Delta S \) during a small time \( \Delta t \) is approximately \( f \Delta S \Delta t \). The total flow of fluid out from the region enclosed by the surface per unit time is the surface integral

\[
\iint_S f \, dS = \lim_{\Delta S_{ij} \to 0} \sum_{i,j} f(x_{ij},y_{ij},z_{ij}) \Delta S_{ij}
\]

where the sum is over a partition of \( S \) into smaller surface areas \( \Delta S_{ij} \), \((x_{ij},y_{ij},z_{ij})\) is any point in \( \Delta S_{ij} \) and we take the limit as the partition becomes finer.

Let us now calculate the rate of flow of fluid \( f \) out per unit area and unit time, given the velocity vector field of the fluid \( \mathbf{V} \) and the density \( \mu \). We define the flow rate density by

\[
\mathbf{F} = \mu \mathbf{V},
\]

If \( \Delta S \) is a small area of a piece of a plane with outward unit normal \( \mathbf{n} \) then we claim that the flow rate out of \( \Delta S \) per unit time is given by

\[
\mathbf{F} \cdot \mathbf{n} \Delta S
\]

In fact, in a small time \( \Delta t \), the fluid particles that will reach \( \Delta S \) are at most \( \mathbf{V} \Delta t \) away, and all particles within reach form a sloped cylinder with \( \Delta S \) as its base and height \( \mathbf{V} \cdot \mathbf{n} \Delta t \). Since the volume is the area of the base times the height the amount of fluid in the cylinder is the density times the volume: \( \mu \mathbf{V} \cdot \mathbf{n} \Delta t \Delta S \). If we divide by \( \Delta t \) we get the rate per unit time and if we divide this by \( \Delta S \) we get the flow rate out of per unit surface area and unit time

\[
f = \mathbf{F} \cdot \mathbf{n}
\]

The flow rate of fluid out of the total surface \( S \), or the flux of the velocity vector field \( \mathbf{F} \) out of the surface \( S \), with outward unit normal \( \mathbf{n} \), is the surface integral

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS
\]
Examples of calculating flow integrals

**Ex** Let $R$ be the 3-dimensional region $R = \{x^2/4 + y + z^2/4 \leq 1, y \geq 0\}$. Let $S$ be the surface of $R$ with the normal oriented outwards. Note that $S$ has two parts $S_1 = \{y = 0, x^2/4 + z^2/4 \leq 1\}$ and $S_2 = \{x^2/4 + y + z^2/4 = 1, y \geq 0\}$.

a) Find the area of $S$.

b) Find the flux of $F = x\mathbf{i} - y\mathbf{j} + \mathbf{k}$ through $S$: $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$.

**Sol** (a) The area of $S_1$ is $4\pi$.

$S_2$ can be viewed as a graph $y = g(x, z) = 1 - x^2/4 - z^2/4$ over the disc $D = \{(x, z); x^2/4 + z^2/4 \leq 1\}$ in the $xz$-plane. With $G(x, y, z) = y - g(x, z)$ the unit normal is

$$n = \frac{\nabla G}{|\nabla G|} = \frac{-gx\mathbf{i} - gz\mathbf{k} + \mathbf{j}}{\sqrt{1 + g_x^2 + g_z^2}} = \frac{x\mathbf{i}/2 + zk/2 + \mathbf{j}}{\sqrt{1 + x^2/4 + z^2/4}}.$$

Now $dS = \frac{dxdz}{\mathbf{n} \cdot \mathbf{j}} = \frac{\sqrt{1 + (x^2 + z^2)/4} \, dxdz}{\sqrt{1 + x^2/4 + z^2/4}}$.

Introducing polar coordinates in the $xz$-plane:

$$\iint_{S_2} dS = \iint_D \left(1 + \frac{x^2 + z^2}{4}\right)^{1/2} \, dxdz = \int_0^2 \int_{\theta=0}^{\pi/2} \left(1 + \frac{r^2}{4}\right)^{1/2} \, d\theta \, dr = 2\pi \left(1 + \frac{r^2}{4}\right)^{3/2} \bigg|_0^2 = \frac{8\pi}{3} \left(2^{3/2} - 1\right)$$

(b) The normal to $S_1$ is $\mathbf{n} = -\mathbf{j}$ and $\mathbf{F} \cdot \mathbf{n} = y$ so the integral over $S_1$ vanishes.

Since $\mathbf{F} \cdot \mathbf{n} = (x^2/2 + z/2 - y)/\sqrt{1 + x^2/4 + z^2/4}$ we obtain

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D (x^2/2 + z/2 - y) \, dxdz = \iint_D (x^2/2 + z/2 - (1 - (x^2 + z^2)/4)) \, dxdz$$

Introducing polar coordinates in the $xz$-plane: $x = r \cos \theta, z = r \sin \theta, dxdz = r \, dr \, d\theta$;

$$\int_0^2 \int_{\theta=0}^{\pi/2} \frac{r^2}{4}(2 \cos^2 \theta + 1) + \frac{r}{2} \sin \theta - 1 \, d\theta \, dr = \int_0^2 \int_{\theta=0}^{\pi/2} \frac{r^2}{4}(2 \cos^2 \theta + 2) - \frac{r}{2} \sin \theta - 1 \, d\theta \, dr = \ldots = 0$$

Alternative solution: Let $X(x, z) = x\mathbf{i} + g(x, z)\mathbf{j} + zk$ and

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = \pm \int_{x^2 + z^2 \leq 4} \mathbf{F} \cdot (X_x \times X_z) \, dxdz = \ldots$$
7.3 Gauss’ Theorem or The divergence theorem. states that if $D$ is a volume bounded by a surface $S$ with outward unit normal $n$ and $F = F_1 i + F_2 j + F_3 k$ is a continuously differentiable vector field in $D$ then

$$\int_S F \cdot n \, dS = \iiint_D \text{div} \, F \, dV,$$

where $\text{div} \, F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

A one dimensional analogue is the First Fundamental Theorem of Calculus:

$$f(b) - f(a) = \int_a^b f'(x) \, dx.$$

A two dimensional analogue says that if $D$ is a region in the plane with boundary curve $C$ and $n = (n_1, n_2)$ is the outward unit normal to $C$, then

$$\int_C F_1 n_1 + F_2 n_2 \, ds = \iint_D \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) \, dA,$$

where $ds$ is the arclength. (This is in fact equivalent to Green’s Theorem with $F_1 = Q$ and $F_2 = -P$ and $n = (dy/ds, -dx/ds)$, where $T = (dx/ds, dy/ds)$ is the unit tangent vector to the boundary curve, which follows since then $T \cdot n = 0$.)

**Ex.** Find flux of $F = 2x i + y^2 j + z^2 k$ out of the unit sphere $S$.

**Sol.** By the divergence theorem we have with $B$ the unit ball

$$\int_S F \cdot n \, dS = \iiint_B \text{div} \, F \, dV = \iiint_B (2 + 2y + 2z) \, dV = \iiint_B 2 \, dV + \iiint_B 2y \, dV + \iiint_B 2z \, dV = 2 \text{Vol}(B) + 0 + 0 = 2 \frac{4\pi}{3}$$

since the last two integrals vanishes because the region is symmetric under replacing $y$ by $-y$ (respectively $z$ by $-z$) but the integrand changes sign.

**Ex.** Find flux of $F = x i + y j - 2z k$ out of the unit sphere.

**Sol.** By the divergence theorem the flux is equal to the integral of the divergence over the unit ball. Since $\text{div} \, F = 0$ it follows that the volume integral vanishes and by the divergence theorem the flux therefore vanishes.

**Physical Interpretation of Divergence** Let $B_\varepsilon$ be a ball of radius $\varepsilon$ and let $S_\varepsilon$ be its surface. Then

$$\int_{S_\varepsilon} F \cdot n \, dS = \iiint_{B_\varepsilon} \text{div} \, F \, dV.$$

By the **mean value theorem for integrals** the right hand side is equal to the volume of the ball $B_\varepsilon$ times $\text{div} \, F$ at some point in the ball so

$$\text{Flux of } F \text{ out through } S_\varepsilon = \text{Vol}(B_\varepsilon) \text{ div } F,$$

where $\text{div} \, F$ is evaluated at some point in $B_\varepsilon$. Hence

$$\text{div} \, F = \lim_{\varepsilon \to 0} \frac{\text{Flux of } F \text{ out through } S_\varepsilon}{\text{Vol}(B_\varepsilon)}.$$
Proof of the divergence theorem for convex sets.

We say that a domain $D$ is convex if for every two points in $D$ the line segment between the two points is also in $D$, e.g. any sphere or rectangular box is convex.

We will prove the divergence theorem for convex domains $D$. Since $F = F_1i + F_2j + F_3k$ the theorem follows from proving the theorem for each of the three vector fields $F_1i$, $F_2j$ and $F_3k$ separately. The theorem for the vector field $F_3k$ states that

$$\iiint_S (F_3k) \cdot n \, dS = \iiint_D \frac{\partial F_3}{\partial z} \, dV$$

Since $D$ is convex we can write $D = \{(x, y, z); f_1(x, y) \leq z \leq f_2(x, y), (x, y) \in D\}$.

Then $S$ consists of two parts $S_1 = \{(x, y, z); z = f_1(x, y), (x, y) \in D\}$ and $S_2 = \{(x, y, z); z = f_2(x, y), (x, y) \in D\}$. We have

$$\iiint_S \frac{\partial F_3}{\partial z} \, dz \, dx \, dy = \iiint_D \int_{f_1(x, y)}^{f_2(x, y)} \frac{\partial F_3}{\partial z} \, dz \, dx \, dy = \int_D F_3((x, y, f_1(x, y))) \, dx \, dy - \int_D F_3((x, y, f_2(x, y))) \, dx \, dy$$

We know that $dx \, dy = k \cdot n \, dS$ on $S_2$ and $dx \, dy = -k \cdot n \, dS$ on $S_1$ so (5.1.4) follows.

**Ex. Gauss law** Let $F = -(xi + yj + zk)/(x^2 + y^2 + z^2)^{3/2}$. Let $S_a$ be the sphere of radius $a$ centered at the origin. Find the flux of $F$ out of $S_a$.

**Sol.** A calculation shows that $\text{div } F = ... = 0$, when $|x| \neq 0$. Hence by the divergence theorem the flux is $\iint_{S_a} F \cdot n \, dS = \iiint_{B_a} \text{div } F \, dV$, where $B_a$ is the ball of radius $a$ centered at 0. From this we deduce that the flux is 0 but this is wrong!

In fact the outward unit normal to $S_a = \{(x, y, z); x^2 + y^2 + z^2 = a^2\}$ is $n = (xi + yj + zk)/(x^2 + y^2 + z^2)^{1/2}$. It therefore follows that

$$\iint_{S_a} F \cdot n \, dS = \iint_{S_a} \frac{dS}{x^2 + y^2 + z^2} = \iint_{S_a} \frac{1}{a^2} \, dS = \frac{1}{a^2} \text{Area}(S_a) = \frac{1}{a^2} 4\pi a^2 = 4\pi$$

Therefore we conclude that the divergence theorem is not valid in this case. In fact $F$ has to be continuously differentiable and bounded in $D$ for the theorem to hold, but $F$ is unbounded at the origin. We also remark that the flux of $F$ out of any region that contains the origin is $4\pi$, and 0 if the region does not contain the origin.

**Mean value theorem for integrals** If $f$ is continuous in $D$ then

$$\iiint_D f \, dV = f(x_0, y_0, z_0) \text{Volume}(D), \text{ for some } (x_0, y_0, z_0) \in D.$$ 

In fact if $\min f \leq f(x, y, z) \leq \max f$ for $(x, y, z) \in D$ so the integral is between $\min f \iiint_D dV \leq \iiint_D f \, dV \leq \max f \iiint_D dV$. Since $f$ is continuous there is $(x_0, y_0, z_0) \in D$ so it is equal to $f(x_0, y_0, z_0) \iiint_D dV$, where $\iiint_D dV = \text{Volume}(D)$. 
