Lecture 20: 7.3 Stokes’ theorem. Let $S$ be a surface with unit normal $n$ and positively oriented boundary $C$, i.e. if you walk in the direction of the curve on the side of the normal then the surface should be on your left. Stokes’ theorem says

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl} \mathbf{F} \cdot n \, dS$$

if $\mathbf{F}$ is a smooth vector field on $S$.

If $S$ is a domain in the $x$-$y$ plane then Stoke’s theorem reduces to Green’s theorem.

In fact

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C P \, dx + Q \, dy,$$

if $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ and

$$\text{curl} \mathbf{F} \cdot n = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

if $n = \mathbf{k}$.

Ex. Find the integral

$$\int_C -y^3 \, dx + x^3 \, dy - z^3 \, dz,$$

where $C$ is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of $C$ corresponds to a counterclockwise motion in the $x$-$y$ plane.

Sol. Let $\mathbf{F} = -y^3 \mathbf{i} + x^3 \mathbf{j} - z^3 \mathbf{k}$. The integral is by Stokes theorem equal to the surface integral of $\text{curl} \mathbf{F} \cdot n$ over some surface $S$ with the boundary $C$ and with unit normal positively oriented with respect to the orientation of the boundary.

We have $\text{curl} \mathbf{F} = ... = (3x^2 + 3y^2) \mathbf{k}$. We take $S$ to be the region in the plane $h(x,y,z) = x + y + z = 1$ with boundary $C$. A unit normal to $S$ is given by $n = \nabla h/|\nabla h| = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$ and it has the correct orientation since $n \cdot \mathbf{k} = 1/\sqrt{3} > 0$.

We therefore get

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl} \mathbf{F} \cdot n \, dS = \iint_S 3(x^2 + y^2)/\sqrt{3} \, dS.$$

Writing $dS = dx \, dy/|n \cdot \mathbf{k}| = \sqrt{3} \, dx \, dy$ we get

$$\iint_{x^2+y^2 \leq 1} 3(x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \int_0^1 3r^2 \, r \, dr \, d\theta = \int_0^{2\pi} \frac{3}{4} r^4 \, d\theta = \frac{3}{4} 2\pi = \frac{3\pi}{2}.$$

Sol. 2. Directly calculating the line integral. Parameterizing the curve $C$ we can write $x = \cos t$, $y = \sin t$ and $z = 1 - x - y = 1 - \cos t - \sin t$, $0 \leq t \leq 2\pi$ and write

$$\int_C -y^3 \, dx + x^3 \, dy - z^3 \, dz = \int_0^{2\pi} \left( -y^3 \frac{dx}{dt} + x^3 \frac{dy}{dt} - z^3 \frac{dz}{dt} \right) \, dt = \int_0^{2\pi} \left( \sin^4 t + \cos^4 t + (1 - \cos t - \sin t)^3 \sin t - (1 - \cos t - \sin t) \right) \, dt = \ldots \text{a lot more work}.$$
**Interpretation of curl.** Furthermore, Stokes Theorem can alternatively be used to define the curl: The component of \( \text{curl } F \) in the direction of a unit vector \( n \) is defined to be the limit as \( \varepsilon \to 0 \) of the line integral of \( F \) around a small circle \( C_\varepsilon \) of radius \( \varepsilon \) perpendicular to \( n \), divided by the area of the disc \( S_\varepsilon \) enclosed by \( C_\varepsilon \):

\[
\int_{C_\varepsilon} F \cdot ds = \iint_{S_\varepsilon} \text{curl } F \cdot n \, dS = \text{curl } F \cdot n \, \text{Area}(S_\varepsilon)
\]

where \( \text{curl } F \cdot n \) is evaluated at some point on \( S_\varepsilon \). It follows that

\[
\text{curl } F \cdot n = \lim_{\varepsilon \to 0} \frac{\int_{C_\varepsilon} F \cdot ds}{\text{Area}(S_\varepsilon)}
\]

**Ex.** Show that \( \int_{C} ye^zdx + xe^zdy + xye^zdz = 0 \) for a closed curve \( C \).

**Sol.** \( F = \nabla(xy^2) \) so \( \text{curl } F = 0 \) and by Stokes’s theorem the integral vanishes.

**Ex.** Find \( \int_{C_a} F \cdot ds \), where \( F = (-yi + xj)/(x^2 + y^2) \) and \( C_a \) is the circle \( x^2 + y^2 = a^2 \) in the \( x\)-\( y \) plane going counterclockwise.

**Sol.** \( \text{curl } F = ... = 0 \). Hence one would have thought that by Stokes theorem the line integral would vanish. **Wrong!** because \( F \) is not continuous.

However, if we parameterize \( x = a \cos t \) and \( y = a \sin t, 0 \leq t < 2\pi \), we get

\[
\int_{C_a} F \cdot ds = \int_{0}^{2\pi} \left( \frac{-y}{x^2 + y^2} \frac{dx}{dt} + \frac{x}{x^2 + y^2} \frac{dy}{dt} \right) dt = \int_{0}^{2\pi} \left( \frac{-a \sin t(-a \sin t)}{a^2} + \frac{a \cos t(a \cos t)}{a^2} \right) dt = \int_{0}^{2\pi} \sin^2 t + \cos^2 t \, dt = 2\pi
\]

The reason Stokes’ theorem failed to hold in this case was that the vector field \( F \) is singular when \( (x, y) = (0, 0) \), i.e. along the \( z \)-axis.
Proof of Stokes’ theorem for a graph. We have seen that Stokes’ theorem for a surface \( S \) in the \( x-y \) plane reduces to Green’s theorem. We will now show that Stokes’ theorem for a surface \( S \) that can be written as a graph \( z = f(x,y) \), \((x,y) \in D\), over a region \( D \) in the plane, also reduces to Green’s theorem. If \( T \) is the tangent vector to the boundary curve \( C \) the Stokes’ theorem can be written:

\[
\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS
\]

where \( ds \) is the arc length and \( dS \) the surface area element. The surface integral can then be written as an integral over \( D \) and the integral over the boundary curve can be written as an integral over the projection of the curve in the \( x-y \) plane. Then one can use Green’s theorem in the plane to show that these things are equal.

Since the surface can be written \( k(x,y,z) = z - f(x,y) \) a normal is given by \( \mathbf{N} = \nabla h = -f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k} \) and the unit normal is given by \( \mathbf{n} = \mathbf{N}/|\mathbf{N}| \). The surface measure is \( dS = dx\,dy/|\mathbf{n}| \), where \( \mathbf{k} \cdot \mathbf{n} = k \cdot \mathbf{N}/|\mathbf{N}| = 1/|\mathbf{N}| \), so \( dS = |\mathbf{N}| \, dx\,dy \) and

\[
\iint_S \mathbf{G} \cdot \mathbf{n} \, dS = \iint_D G_1 f_x - G_2 f_y + G_3 \, dx\,dy, \quad \text{if} \quad \mathbf{G} = G_1 \mathbf{i} + G_2 \mathbf{j} + G_3 \mathbf{k}
\]

If we apply to \( \mathbf{F} \) this to \( \mathbf{G} = \text{curl} \mathbf{F} \) we get

\[
\iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D -\left( \frac{\partial F_3}{\partial y} \right) f_x - \left( \frac{\partial F_1}{\partial z} \right) f_y + \left( \frac{\partial F_2}{\partial x} \right) f_z \, dx\,dy.
\]

If we parameterize the boundary \( x = x(t) \), \( y = y(t) \) and \( z = f(x,y) \) we have

\[
\frac{dz}{dt} = f_z \frac{dx}{dt} + f_y \frac{dy}{dt},
\]

\[
\int_C \mathbf{F} \cdot ds = \int_a^b \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt = \int_a^b \left( (F_1 + f_z f_3) \frac{dx}{dt} + (F_2 + f_y f_3) \frac{dy}{dt} \right) dt
\]

This can now be considered as a line integral in the plane:

\[
\int_C \mathbf{F} \cdot ds = \oint_{\partial D} P \, dx + Q \, dy, \quad \text{where} \quad P(x,y) = F_1(x,y,f(x,y)) + f_x(x,y)f_3(x,y,f(x,y)),
\]

\[
Q(x,y) = F_2(x,y,f(x,y)) + f_y(x,y)f_3(x,y,f(x,y))
\]

We can therefore apply Greens formula in the plane.

\[
\frac{\partial P}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_3}{\partial y} f_y + f_x \frac{\partial F_3}{\partial y}, \quad \frac{\partial Q}{\partial x} = \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial x} f_x + f_y \frac{\partial F_3}{\partial x}
\]

so by Green’s theorem

\[
\int_C \mathbf{F} \cdot ds = \oint_{\partial D} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx\,dy
\]

\[
= \iint_D \left( \frac{\partial F_3}{\partial y} \right) f_x - \left( \frac{\partial F_1}{\partial z} \right) f_y + \left( \frac{\partial F_2}{\partial x} \right) f_z \, dx\,dy,
\]
Some more calculations of surface integrals.

Ex. Let $S$ be the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ with $0 \leq z \leq 1$.

A parametrization of the surface is given by

$$X(u, v) = (\cos u - v \sin u)i + (\sin u + v \cos u)j + vk, \quad 0 \leq u \leq 2\pi, \quad 0 \leq v \leq 1.$$ 

a) Find the area element $dS$ expressed in terms of the parametrization $du \, dv$.

b) Find the surface integral $\int_S zdS$.

Sol. a) $X_u = (-\sin u - v \cos u)i + (\cos u - v \sin u)j$ and $X_v = -\sin u + \cos u j + k$;

$$X_u \times X_v = \begin{vmatrix} i & j & k \\ -\sin u - v \cos u & \cos u - v \sin u & 0 \\ -\sin u & \cos u & 1 \end{vmatrix} = (\cos u - v \sin u)i + (\sin u + v \cos u)j - vk$$ 

Hence $dS = |X_u \times X_v| \, du \, dv = \sqrt{1 + 2v^2} \, du \, dv$.

b) $\int \int_S z \, dS = \int_0^{2\pi} \int_0^1 v \sqrt{1 + 2v^2} \, du \, dv = 2\pi \int_0^1 \sqrt{1 + 2v^2} \, dv = \pi \left[ \frac{(1 + 2v^2)^{3/2}}{3} \right]_0^1 = \frac{\pi}{3} \left( 3^{3/2} - 1 \right)$.