Lecture 5: Section 2.3 Cont.. If \( F : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is given by \( F(x) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3)) \), then the derivative matrix of partial derivatives is defined by

\[
DF = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \frac{\partial F}{\partial x_3}
\end{bmatrix}
\]

Here the second expression mean that the we think of \( F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \) as a column vector and \( \frac{\partial F}{\partial x_1} \) as the derivative of the column vector. Similarly we define the \( m \times n \) matrix \( DF \) for a function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \).

A special case is the gradient of a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) given by

\[
\text{grad } f = \nabla f = \begin{bmatrix}
\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n}
\end{bmatrix}
\]

A function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is called differentiable at \( a \) if

\[
\lim_{x \to a} \frac{\|F(x) - F(a) - DF(a)(x - a)\|}{\|x - a\|} = 0
\]

Here \( DF(a)(x - a) \) is matrix multiplication of the \( m \times n \) matrix or column vector \( DF(a) \) by the \( n \times 1 \) matrix \( x - a \). If \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) then

\[
Df(a)(x - a) = \begin{bmatrix}
\frac{\partial f}{\partial x_1}(a) & \frac{\partial f}{\partial x_2}(a)
\end{bmatrix} \begin{bmatrix}
x_1 - a_1 \\
x_2 - a_2
\end{bmatrix} = \frac{\partial f}{\partial x_1}(a)(x_1 - a_1) + \frac{\partial f}{\partial x_2}(a)(x_2 - a_2)
\]

If \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) then

\[
DF(a)(x - a) = \begin{bmatrix}
\frac{\partial F}{\partial x_1}(a) & \cdots & \frac{\partial F}{\partial x_n}(a)
\end{bmatrix} \begin{bmatrix}
x_1 - a_1 \\
\vdots \\
x_n - a_n
\end{bmatrix} \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(a) \\
\vdots \\
\frac{\partial f_m}{\partial x_n}(a)
\end{bmatrix} (x_1 - a_1) + \cdots + \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(a) \\
\vdots \\
\frac{\partial f_m}{\partial x_n}(a)
\end{bmatrix} (x_n - a_n)
\]

We can think of the derivative of \( F \) at the point \( a \in \mathbb{R}^n \) as the linear map \( DF : \mathbb{R}^n \rightarrow \mathbb{R}^m \), mapping the vector \( h = (h_1, ..., h_n) \) to the directional derivative

\[
DF(a)h = \lim_{t \to 0} \frac{F(a + th) - F(a)}{t} = \frac{\partial F}{\partial x_1}(a)h_1 + ... + \frac{\partial F}{\partial x_n}(a)h_n.
\]

Ex If \( A \) is an \( m \times n \) matrix we define \( F(x) = Ax : \mathbb{R}^n \rightarrow \mathbb{R}^m \). Then \( DF = A \).
Th If $f$ is differentiable then it is continuous.
Pf See book.

Th If the partial derivatives are continuous in a neighborhood that the function is differentiable in the neighborhood.
Pf See book.

Section 2.4.
Th $\partial_x \partial_y f(x, y) = \partial_y \partial_x f(x, y)$
Pf See book.

We also went over Newton’s method for finding an approximation of a solution to $f(x) = 0$. 