Lecture 9: 4.1 Taylor’s formula in several variables.

Recall Taylors formula for $f : \mathbb{R} \rightarrow \mathbb{R}$:

1. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x-a, a)$

where the remainder or error tends to 0 faster than the previous terms when $x \rightarrow a$:

2. $|R_k(x-a, a)| \leq \frac{M}{(k+1)!}|x-a|^{k+1}, \quad \text{if} \quad |f^{(k+1)}(z)| \leq M,$

for $|z-a| \leq |x-a|$. The Taylor polynomial $P_k = f_k - R_k$ is the polynomial of degree $k$ that best approximate $f(x)$ for $x$ close to $a$. It is chosen so its derivatives of order $\leq k$ are equal to the derivatives of $f$ at $a$. (2) follows from repeated integration of

\begin{equation}
\frac{d^{k+1}}{dx^{k+1}} R_k(x-a, a) = f^{(k+1)}(x), \quad \frac{d^j}{dx^j} R_k(x-a, a) \bigg|_{x=a} = 0, \quad j \leq k.
\end{equation}

A similar formula hold for functions of several variables $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$. In order to state it we first write $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$.

3. $F(x) = F(a) + \sum_{i=1}^n F_{x_i}(a)(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^n F_{x_i x_j}(a)(x_i - a_i)(x_j - a_j)$

\[ \ldots + \frac{1}{k!} \sum_{i_1, \ldots, i_k=1}^n F_{x_{i_1} \ldots x_{i_k}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_k}) + R_k(x-a, a), \]

where the remainder or error tends to 0 faster than the previous terms when $x \rightarrow a$:

4. $\|R_k(x,a)\| \leq \frac{M}{(k+1)!}\|x-a\|^{k+1}, \quad \text{if} \quad \sum_{i_1, \ldots, i_{k+1}=1}^n \|F_{x_{i_1} \ldots x_{i_{k+1}}}(z)\| \leq M,$

for $\|z-a\| \leq \|x-a\|$. Here

\begin{equation}
F_{x_{i_1} \ldots x_{i_k}} = \frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}
\end{equation}

First, the general case reduces to the case $m = 1$ by considering each component of $F = (F_1, \ldots, F_m)$ and we may hence assume that $F : \mathbb{R}^n \rightarrow \mathbb{R}$. In order to prove (3) we introduce $x - a = h$ and apply the one dimensional Taylor’s formula (1) to the function $f(t) = F(x(t))$ along the line segment $x(t) = a + th, 0 \leq t \leq 1$:

\begin{equation}
\frac{d}{dt} f(t) = f(0) + f'(0) + \frac{f''(0)}{2} + \ldots + \frac{f^{(k)}(0)}{k!} + R_k
\end{equation}

Here $f(1) = F(a + h)$, i.e. the left hand side of (3), $f(0) = F(a)$, i.e. the first term in the right hand side of (3), and by the chain rule

\begin{equation}
f'(t) = \frac{d}{dt} F(x(t)) = \sum_{i=1}^n F_{x_i}(x(t)) \frac{dx_i}{dt} = \sum_{i=1}^n F_{x_i}(x(t)) h_i
\end{equation}

and hence $f'(0)$ is the second term in the right of (3). Repeating this gives

\begin{equation}
f''(t) = \frac{d}{dt} \sum_{i=1}^n F_{x_i}(x(t)) h_i = \sum_{i=1, j=1}^n F_{x_i x_j}(x(t)) h_i h_j
\end{equation}

and this gives the third term and so on.
If \( f : \mathbb{R}^2 \to \mathbb{R} \), \( a = (0, 0) \) and \( x = (x, y) \) then the second degree Taylor polynomial is
\[
f(x, y) \sim f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}(f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2)
\]
Here we used the equality of mixed partial derivatives \( f_{xy} = f_{yx} \).

**Ex.** Let \( f(x, y) = 3 + 2x + x^2 + 2xy + 3y^2 + x^3 - y^4 \). Find the second degree Taylor polynomial around \( a = (0, 0) \).

**Sol.** The second degree Taylor polynomial is
\[
f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}(f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2)
\]
\[
= 3 + 2x + \frac{1}{2}(2x^2 + 2 \cdot 2xy + 6y^2)
\]
Please note that \( f_{xy} = f_{yx} \), so we have equality of mixed partial derivatives.

**The derivative of a vector field as a linear map.** Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be a vector field. Then we can think of the derivative of \( F \) as a linear map. Let \( \mathbf{h} = (h_1, ..., h_n) \) be the vector field.

**Def** The **Hessian** of a function \( f : \mathbb{R}^n \to \mathbb{R} \) is the \( n \times n \) matrix
\[
Hf = \begin{bmatrix}
f_{x_1x_1} & \cdots & f_{x_1x_n} \\
\vdots & \ddots & \vdots \\
f_{x_nx_1} & \cdots & f_{x_nx_n}
\end{bmatrix}
\]

The second order Taylor formula for a function \( f : \mathbb{R}^n \to \mathbb{R} \) can hence be written:
\[
f(x) = f(a) + Df(a)(x - a) + \frac{1}{2}(x - a)^T Hf(a)(x - a) + R_2(a, x)
\]
where \( T \) stands for transpose; \( (x - a)^T Hf(a)(x - a) = (x - a) \cdot Hf(a)(x - a) \).