Lecture 10: 3.3 Complex roots. In this chapter we want to solve the equation

$$ay'' + by' + cy = 0 \tag{3.3.1}$$

where a, b, c are real constants. We saw that $y = e^{rt}$ is a solution to the equation if r is a root of the characteristic equation:

$$ar^2 + br + c = 0 \tag{3.3.2}$$

Let r_1 and r_2 be the roots of (3.3.2). In section 3.1 we saw that if $r_1 \neq r_2$ and they are both real the general solution of (3.3.1) is in fact of the form $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for some constants c_1 and c_2. We will now consider the case of complex roots and since a, b, c are real the complex roots come in complex conjugate pairs (show this) so unless r_1 and r_2 both are real they must be of the form

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \qquad i = \sqrt{-1} \tag{3.3.3}$$

Analogous to the real case we hope that we get to solutions of the form

$$y_1 = e^{r_1 t}, \qquad y_2 = e^{r_2 t} \tag{3.3.4}$$

However, we don't even know what e^z is supposed to mean when z is complex. In order for (3.3.4) to be solutions for complex numbers we must be able to extend the definition of the exponential functions to complex numbers in such a way that

$$\frac{d}{dt} e^{rt} = r e^{rt} \tag{3.3.5}$$

also if r is complex. To get a clue we must ask what it is when z is real. One expression is the Taylor series

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{3.3.6}$$

This makes sense also when z is complex since we defined how to multiply and add complex numbers and since the sum is absolutely convergent. It is easy to see that (3.3.5) follows by differentiating the series (3.3.6) termwise:

$$\frac{d}{dt} \sum_{k=0}^{\infty} \frac{z^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{z^k t^{(} k-1)}{(k-1)!} = z \sum_{k=0}^{\infty} \frac{z^k t^k}{k!}$$

However, if we were to use (3.3.6) as a definition we must prove that $e^{z_1 + z_2} = e^{z_1} e^{z_2}$ also for complex numbers z_1 and z_2 and showing that we get the series for the sum when we multiply together the series would involve proving some combinatorial identities. Instead we use (3.3.6) to get an expression we take as definition. By (3.3.6);

$$e^{i\mu} = \sum \frac{(i\mu)^k}{k!} = \sum \frac{(-1)^n \mu^{2n}}{(2n)!} + i \sum \frac{(-1)^{n-1} \mu^{2n-1}}{(2n-1)!}$$

But the two series are the Taylor series for $\cos\mu$ respectively $\sin\mu$ so we get

$$e^{i\mu} = \cos\mu + i\sin\mu.$$

Since we want the product rule $e^{\lambda+i\mu} = e^{\lambda}e^{i\mu}$ to hold we define

$$e^{\lambda+i\mu} = e^{\lambda}(\cos\mu + i\sin\mu)$$

for any complex number $\lambda + i\mu$ and hence also for any t

$$e^{rt} = e^{(\lambda+i\mu)t} = e^{\lambda t}\big(\cos(\mu t) + i\sin(\mu t)\big), \qquad r = \lambda + i\mu$$

With this definition it follows from differentiation that also for complex r

$$\frac{d}{dt}\, e^{rt} = r\, e^{rt}$$

Since this was the rule used to prove that (3.3.4) are solutions of (3.3.1) when (3.3.3) are the roots of (3.3.2) it follows that indeed (3.3.4) are solutions to (3.3.1) also when r_1 and r_2 are complex roots of the characteristic polynomial.

We have now found two solution to (3.3.1)

$$(3.3.6) \qquad z_1 = e^{\lambda t}\big(\cos(\mu t) + i\sin(\mu t)\big), \qquad\qquad z_2 = e^{\lambda t}\big(\cos(\mu t) - i\sin(\mu t)\big)$$

In fact the expression for z_2 follows from the one for z_1 by replacing μ by $-\mu$ and using that $\cos(-\mu t) = \cos(\mu t)$ and $\sin(-\mu t) = -\sin(\mu t)$. There is however one remaining problem which is that (3.3.6) are complex but we expect the solutions to (3.3.1) to correspond to some real physical quantity. However

$$(3.3.7) \qquad y_1 = \frac{z_1 + z_2}{2} = e^{\lambda t}\cos(\mu t), \qquad\qquad y_2 = \frac{z_1 - z_2}{2i} = e^{\lambda t}\sin(\mu t)$$

are real solutions to (3.3.1). We can now if we want forget the derivation of these solutions using complex numbers and instead just check that they are solutions. We claim that the general solution of (3.3.1) is of the form

$$y = c_1 y_1 + c_2 y_2$$

In fact, by section 3.2 we only need to check that the Wronskian is non-vanishing

$$W = y_1 y_2' - y_1' y_2 = e^{\lambda t}\cos(\mu t) e^{\lambda t}\big(\mu\cos(\mu t) + \lambda\sin(\mu t)\big)$$
$$-e^{\lambda t}\big(-\mu\sin(\mu t) + \lambda\cos(\mu t)\big)e^{\lambda t}\sin(\mu t) = e^{2\lambda t}\big(\cos^2(\mu t) + \sin^2(\mu t)\big) = e^{2\lambda t} \neq 0$$

Ex Find all solutions to the equation

$$y'' + 2y' + 5y = 0$$

Sol The characteristic polynomial is

$$r^2 + 2r + 5 = (r + 1 + 2i)(r + 1 - 2i)$$

with roots $r_1 = -1 + 2i$ and $r_2 = -1 - 2i$ so the general solution is

$$y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$