Lecture 11: 3.4: Repeated roots. In this chapter we want to solve the equation

$$L[y] \equiv ay'' + by' + cy = 0 \tag{3.4.1}$$

where a, b, c are real constants. We saw that $y = e^{rt}$ is a solution to the equation if r is a root of the characteristic equation:

$$ar^2 + br + c = a(r - r_1)(r - r_2) = 0 \tag{3.4.2}$$

We have dealt with distinct real roots $r_1 \neq r_2$ and complex conjugate roots, $r_2 = \overline{r}_1$. It only remains to deal with the case of double roots: $r_1 = r_2 = r = -\frac{b}{2a}$. In that case $e^{r_1 t}$ and $e^{r_2 t}$ is the same function so we only have one independent solution

$$y_1 = e^{rt}. \tag{3.4.3}$$

and we must obtain another solution. It is easy to check that if $r_1 = r_2 = r$ then

$$y_2 = t\, e^{rt} \tag{3.4.4}$$

is another solution to (3.4.1) and $W(y_1, y_2) \neq 0$ so any solution can be written as

$$y = c_1 y_1 + c_2 y_2. \tag{3.4.5}$$

How did we find the solution y_2? We make the following ansatz

$$y_2 = v y_1$$

We put this into the equation and we will obtain a simpler equation for v. In fact, if v is any function then $(vy_1)' = v'y_1 + vy_1'$ and $(vy_1)'' = v''y_1 + 2v'y_1' + vy_1''$ so

$$\begin{aligned} a(vy_1)'' + b(vy_1)' + cvy_1 &= a\big(v''y_1 + 2v'y_1' + vy_1''\big) + b\big(v'y_1 + vy_1'\big) + cvy_1 \\ &= av''y_1 + v'\big(2ay_1' + by_1\big) + v\big(ay_1'' + by_1' + cy_1\big) = av''y_1 + v'\big(2ay_1' + by_1\big) \end{aligned} \tag{3.4.6}$$

since y_1 is a solution. The importance of this is that its a first order system

$$w' + \frac{\big(2ay_1' + by_1\big)}{ay_1} w = 0, \quad \text{for} \quad v' = w,$$

which can be solved, first for w, then for v. This is called **reduction of order**.

In case of a double root $r = -\frac{b}{2a}$ its even simpler since $y_1' + by_1 = (2ar + b)e^{rt} = 0$.

$$w' = 0 \quad \Leftrightarrow \quad w = C_1 \qquad v' = C_1 \quad \Leftrightarrow \quad v = C_1 t + C_2$$

If we choose $C_1 = 1$ and $C_2 = 0$ we get (3.4.3) and if $C_1 = 0$ and $C_2 = 1$ we get (3.4.4).

Ex Find the general solution to $y'' - 4y' + 4y = 0$.
Sol The characteristic polynomial is $r^2 - 4r + 4 = (r-2)^2$ so it has a double root $r_1 = r_2 = 2$. The general solution is hence $y = c_1 y_1 + c_2 y_2$, where $y_1 = e^{2t}$ and $y_2 = te^{2t}$.

We will explore two other methods that have the advantage that they can be used also for the Nonhomogeneous problem. If $L[y] = (aD^2 + bD + c)y$, where D is the time derivative, and $p(r) = ar^2 + br + c$ then one can prove the following formula:

$$L\big[v(t)e^{rt}\big] = \big(p\,(r)v(t) + p'(r)v'(t) + \frac{1}{2}p''(r)v''(t)\big)e^{rt} \tag{3.4.8}$$

by a direct calculation as above checking that both sides are the same.
If r is a double root then $p(r) = p'(r) = 0$ so (3.4.8) vanishes if $v'' = 0$.
Anther way is to write $L[y] = a(D - r_1)(D - r_2)y = 0$ and solve the two problems

$$(D - r_1)z = 0, \qquad (D - r_2)y = z \tag{3.4.9}$$

The first gives $z = c_1 e^{r_1 t}$ and the second gives after multiplying by an integrating factor $D(e^{-r_2 t}y) = e^{(r_1 - r_2)t}$. If $r_1 = r_2$ we get if we integrate $e^{-r_1 t}y = c_2 t + c_1$ or $y = c_2 t e^{r_1 t} + c_1 e^{r_1 t}$.

3.5: Nonhomogeneous equation. We return to the **nonhomogeneous** case

(3.5.1) $$L[y] \equiv y''' + p(t)y' + q(t)y = g(t)$$

The complementary **homogeneous** equation is when $g(t) \equiv 0$:

(3.5.2) $$L[y] \equiv y''' + p(t)y' + q(t)y = 0$$

Th If Y_1 and Y_2 are solutions to (3.5.1) then $Y_1 - Y_2$ is a solution to (3.5.2).
Pf

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0.$$

Th Let Y be a particular solution to (3.5.1). Then the general solution to (3.5.1) is

(3.5.2) $$y = c_1y_1 + c_2y_2 + Y$$

where y_1 and y_2 is a fundamental set of solutions to (3.5.2).
Pf If $Y_1 = y$ is an arbitrary solution and $Y_2 = Y$ then by the previous theorem $y - Y$ is a solution to (3.5.2) and since y_1 and y_2 form a fundamental set of solutions it must be equal to $c_1y_1 + c_2y_2$ for some constants c_1 and c_2.

The strategy for finding solutions to (3.5.1) is now clear.
(i) Find the most general solution to (3.5.2), called the complementary solution y_c.
(ii) Find one solution Y to (3.5.1), called a particular solution.
(iii) Obtain the most general solutions to (3.5.1) as $y = y_c + Y$.

We now return to the constant coefficient case to investigate methods to find particular solutions to solve certain nonhomogeneous problems:

The method of undetermined coefficients.
Ex Find the most general solution to

(3.5.3) $$L[y] \equiv y'' + 3y' + 2y = e^{4t}$$

Sol The characteristic polynomial is $r^2+3r+2 = (r+2)(r+1)$ so the complimentary solution is $y_c = c_1e^{-t} + c_2e^{-2t}$. We must now find one particular solution. Why don't we try $Y = Ae^{4t}$ since e^{4t} is on the right hand side

$$L[Ae^t] = 16Ae^{4t} + 12Ae^{4t} + 2Ae^{4t} = 30Ae^{4t} = e^{4t}$$

if $A = 1/30$ so $Y = e^{4t}/30$ is a particular solutions. Hence the most general solution is

$$y = c_1e^{-t} + c_2e^{-2t} + \frac{1}{30}e^{4t}$$