Lecture 12: 3.5: Nonhomogeneous equation. We return to the **nonhomogeneous** case

$$L[y] \equiv y''' + p(t)y' + q(t)y = g(t) \tag{3.5.1}$$

The complementary **homogeneous** equation is when $g(t) \equiv 0$:

$$L[y] \equiv y''' + p(t)y' + q(t)y = 0 \tag{3.5.2}$$

Th If Y_1 and Y_2 are solutions to (3.5.1) then $Y_1 - Y_2$ is a solution to (3.5.2).
Pf

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) = g(t) = 0.$$

Th Let Y be a particular solution to (3.5.1). Then the general solution to (3.5.1) is

$$y = c_1 y_1 + c_2 y_2 + Y \tag{3.5.2}$$

where y_1 and y_2 is a fundamental set of solutions to (3.5.2).
Pf If $Y_1 = y$ is an arbitrary solution and $Y_2 = Y$ then by the previous theorem $y - Y$ is a solution to (3.5.2) and since y_1 and y_2 form a fundamental set of solutions it must be equal to $c_1 y_1 + c_2 y_2$ for some constants c_1 and c_2.

The strategy for finding solutions to (3.5.1) is now clear.
(i) Find the most general solution to (3.5.2), called the complementary solution y_c.
(ii) Find one solution Y to (3.5.1), called a particular solution.
(iii) Obtain the most general solutions to (3.5.1) as $y = y_c + Y$.

We now return to the constant coefficient case to investigate methods to find particular solutions to solve certain nonhomogeneous problems:

The method of undetermined coefficients.
Ex Find the most general solution to

$$L[y] \equiv y'' + 3y' + 2y = e^{4t} \tag{3.5.3}$$

Sol The characteristic polynomial is $r^2+3r+2 = (r+2)(r+1)$ so the complimentary solution is $y_c = c_1 e^{-t} + c_2 e^{-2t}$. We must now find one particular solution. Why don't we try $Y = Ae^{4t}$ since e^{4t} is on the right hand side

$$L[Ae^{t}] = 16Ae^{4t} + 12Ae^{4t} + 2Ae^{4t} = 30Ae^{4t} = e^{4t}$$

if $A = 1/30$ so $Y = e^{4t}/30$ is a particular solutions. Hence the most general solution is

$$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{30} e^{4t}$$

Ex Find a particular solution to

$$L[y] \equiv y'' + 3y' + 2y = \sin t \tag{3.5.4}$$

$A\sin t$ will not work because we will also get $3A\cos t$ in the left. Let us instead try $Y = A\sin t + B\cos t$, in which $Y' = A\cos t - B\sin t$ and $Y'' = -A\sin t - B\cos t$ so

$$\begin{aligned} L\big[A\sin t + B\cos t\big] &= -A\sin t - B\cos t + 3A\cos t - 3B\sin t + 2A\sin t + 2B\cos t \\ &= (-A - 3B + 2A)\sin t + (-B + 3A + 2B)\cos t = \sin t \end{aligned}$$

if $-A-3B+2A=A-3B=1$ and $-B+3A+2B=B+3A=0$, which is equivalent to $B=-3A$ and $10A=1$, i.e. $A=1/10$ and $B=-3/10$. The particular solution is

$$Y = \frac{1}{10}\sin t - \frac{3}{10}\cos t$$

The reason it does not work with just $\sin t$ is that $\sin t = (e^{it} - e^{-it})/2i$ and one can separately solve for $e^{it}/2i$ and $-e^{-it}/2i$ and add the results as we shall see below

Two find the solution for more involved right hand sides that are sums we use that

$$L[Y_1] = g_1, \qquad L[Y_2] = g_2 \qquad \Longrightarrow \qquad L[Y_1 + Y_2] = L[Y_1] + L[Y_2] = g_1 + g_2$$

Ex Find a particular solution to

$$y'' + 3y' + 2y = e^t + \sin t \tag{3.5.5}$$

Sol Adding up the solutions in the previous example gives

$$Y = \frac{1}{30}e^{4t} + \frac{1}{10}\sin t - \frac{3}{10}\cos t$$

Ex Find a particular solution to

$$L[y] \equiv y'' + 3y' + 2y = e^{-t} \tag{3.5.3}$$

Sol It doesn't work with Ae^{-t} since e^{-t} is a solution to the homogeneous equation $L[e^{-t}] = 0$. We will therefore try $Y = Ate^{-t}$. Then $Y' = Ae^{-t} - Ate^{-t}$ and $Y'' = -2Ae^{-t} + Ate^{-t}$ so

$$L[Ate^{-t}] = -2Ae^{-t} + Ate^{-t} + 3\big(Ae^{-t} - Ate^{-t}\big) + 2Ate^{-t} = Ae^{-t} = e^{-t},$$

if $A = 1$ so a particular solution is

$$Y = te^{-t}$$

The general rule If $g(t) = ...$ then try with $Y(t) = ...$:

$$\begin{aligned} &g(t) = a_0t^n + ... + a_n, && Y(t) = t^s(A_0t^n + ... + A_n) \\ &g(t) = (a_0t^n + ... + a_n)e^{\alpha t}, && Y(t) = t^s(A_0t^n + ... + A_n)e^{\alpha t} \\ &g(t) = (a_0t^n + ... + a_n)e^{\alpha t}\cos\beta t && Y(t) = t^s(A_0t^n + ... + A_n)e^{\alpha t}\cos\beta t \\ &\qquad +(b_0t^n + ... + b_n)e^{\alpha t}\sin\beta t, && \qquad + t^s(B_0t^n + ... + B_n)e^{\alpha t}\sin\beta t \end{aligned}$$

where $s = 0, 1, 2$ is the smallest integer that will assure that Y is not a solution to the homogeneous equation.