Lecture 14: 3.7 Free vibrations.

Consider a mass m hanging in a spring. The mass causes an elongation L of the spring in the downward (positive) direction. The gravitational force mg acts downwards and there is a balancing upward force F_s, due to the spring. By Hooke's law $F_s = -kL$, where the constant of proportionality k is called the spring constant. If the mass is in equilibrium, i.e. static, then force balance gives $mg - kL = 0$.

We now want to study the dynamic problem of he motion of the mass. Let $u(t)$, measured positive downwards, denote the displacement of the mass from its equilibrium position, at time t. Then by Newton's second law, the mass times the acceleration of the mass is equal to total force acting on the mass:

$$mu'' = mg + F_s + F_d + F$$

Here mg is the gravitational force and $F_s = -k(L+u)$ is the spring force. $F_d = -\gamma u'$ is a force due to damping or friction and F is a possible external force. Since we already calculated that $kL = mg$ these forces cancel each other and we get

$$mu'' = mg - k(L+u) - \gamma u' + F = -ku - \gamma u' + F$$

or

$$mu'' + \gamma u' + ku = F, \qquad k > 0, \gamma \geq 0$$

We furthermore given the mass some initial position and velocity:

$$u(0) = u_0, \qquad u'(0) = v_0$$

Let us first look on **undamped** ($\gamma = 0$) **free** ($F = 0$) **vibrations**:

$$mu'' + ku = 0$$

The characteristic polynomial is $mr^2 + k = 0$ so $r = \pm\omega_0\, i$, where $\omega_0 = \sqrt{k/m}$ so

$$u = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta),$$

where $R = \sqrt{A^2 + B^2}$ is the amplitude and δ, given by $\tan\delta = A/B$, is a phase factor. Note that the frequency ω_0 and period $T = 2\pi/\omega_0$ of the vibration depends only on the spring constant and mass but is independent on initial conditions.

Let us first look on **damped** ($\gamma > 0$ **free** ($F = 0$) **vibrations**:

$$mu'' + \gamma u + ku = 0$$

The characteristic polynomial is $mr^2 + \gamma r + k = 0$ with roots:

$$r_1, r_2 = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{(2m)^2} - \frac{k}{m}}$$

If $\gamma^2 < 4km$ then with $\mu = \sqrt{\gamma^2/(2m)^2 - k/m}$ we get a damped vibration

$$u = e^{-\gamma t/2m}\big(A\cos\mu t + B\sin\mu t\big) = Re^{-\gamma t/2m}\cos(\mu t - \delta),$$

This identity is proved as follows. At the critical damping when $\gamma^2 = 4km$ we get

$$u = (A + Bt)e^{-\gamma t/2m}$$

and when $\gamma^2 > 4km$ we get

$$u = Ae^{r_1 t} + Be^{r_2 t}$$

We get exactly the same equation for an electric circuit as for a spring. Consider the RCL-circuit of a resistor R, a capacitor C and an inductor L coupled in a series circuit with an external voltage source E applied. Then adding up the voltage drops over the components we get with Q denoting the charge and $Q' = I$ the current:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

3.8 Forced vibrations.
Let us now consider the case of forced undamped vibrations:

$$u'' + k\,u = F_0 \cos\omega t$$

Physical examples of this are the electric circuit with a voltage forced on it, mentioned above but also a car attached to spring that can accelerate. The general solution is if $\omega \neq \omega_0 = \sqrt{k/m}$:

$$u = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos\omega t$$

In particular if we choose initial conditions $u(0) = u'(0) = 0$ we get

$$u = \frac{F_0}{m(\omega_0^2 - \omega^2)} \big(\cos\omega t - \cos\omega_0 t \big)$$

Using the formula $\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$ this can also be written as

$$u = \Big(\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2} \Big) \sin\frac{(\omega + \omega_0)t}{2}$$

If ω is very close to ω_0 then $|\omega - \omega_0|/2$ is a small compared to $|\omega + \omega_0|/2$ and one can think of the parenthesis as a slowly varying amplitude. This is used for amplitude modulation radio waves.
Note that as $\omega \to \omega_0$ the amplitude becomes larger and using l'Hospitals rule or the Taylor series for $\sin\alpha \sim \alpha$, we get

$$\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2} \to \frac{F_0}{2m\omega_0} t, \qquad \omega \to \omega_0$$

When $\omega = \omega_0$ we have **resonance**, then the particular solution is no longer given by the above and instead it is

$$u = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F_0}{2m\omega_0} t \sin\omega_0 t$$

In this case we can put in a constant force to a system and the solution builds up over time and becomes larger and larger.