Lecture 16: 6.1-6.2 More Laplace transforms. Recall the Laplace transform:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{6.2.1}$$

This is defined for $s > a$ if f is piecewise continuous and $|f(t)| \leq Ke^{at}$, for $t \geq M$. We need to show that the limit as $T\infty$ of

$$\int_0^T e^{-st} f(t)\, dt = \int_0^M e^{-st} f(t)\, dt + \int_M^T e^{-st} f(t)\, dt$$

exist. The first integral is bounded independent of T and and the second integral can be estimated by

$$\Big|\int_M^T e^{-st} f(t)\, dt\Big| \leq \int_M^T e^{-st} |f(t)|\, dt \leq \int_0^T e^{-st} K e^{at}\, dt = K \int_0^T e^{-(s-a)t}\, dt$$
$$= \frac{-K}{s-a} e^{(s-a)t}\Big|_0^T = \frac{K}{s-a}\big(1 - e^{-(s-a)T}\big) \leq \frac{K}{s-a}$$

For a proof of the inequalities above compare the areas below the graph of the functions. It follows that it can not go to infinity as $T \to \infty$, and one can show using a similar argument that it actually converges.

Inverse Laplace transform. We never actually need to put up a formula for the inverse of the Laplace transform but we only need to know that its invertible. Instead we will use a big table together with properties of the Laplace transform to be able to go backwards from known Laplace transforms. It requires some complex analysis to understand the inversion formula, but it can be reduces to the fact the Fourier transform is invertible. The inversion formula is

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \mathcal{L}_s^{-1}\{F(s)\}(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds,$$

where the integration if along a line with constant real part $\mathrm{Re}(s) = \gamma$ in the complex place, provide that $F(s)$ is an analytic function in the half plane $\mathrm{Re}(s) > \gamma$. If we make a complex change of variables this can be written

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(\gamma + i\xi)t} F(\gamma + i\xi)\, d\xi = e^{\gamma t} \mathcal{F}^{-1}\big(F(\gamma + i\xi)\big),$$

i.e. the inverse Fourier transform of the function $\xi \to F(\gamma + i\xi)$ multiplied by $e^{\gamma t}$.

Last time we showed that:

Ex 1 $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, when $s > a$.

Th 1 $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

Th 2 $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

We will now apply these rules to get more Laplace transforms:

Ex 2 $\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}$, when $s > 0$.

Sol Using Euler's formula $\sin(bt) = \dfrac{1}{2i}\left(e^{bit} - e^{-bit}\right)$ and Ex 1 with a replaced by ib and $-ib$ we get

$$\mathcal{L}\{\sin bt\} = \frac{1}{2i}\mathcal{L}\{e^{bit}\} - \frac{1}{2i}\mathcal{L}\{e^{-bit}\} = \frac{1}{2i}\Big(\frac{1}{s-ib} - \frac{1}{s+ib}\Big)$$
$$= \frac{1}{2i}\,\frac{s+ib-(s-ib)}{(s+ib)(s-ib)} = \frac{1}{2i}\,\frac{2ib}{s^2-(ib)^2} = \frac{b}{s^2+b^2}$$

In the same way, or alternatively using Th 1 one can show that;

Ex 3 $\mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}$, when $s > 0$.

Ex 4 A converse of Th 1 is also hold:

Th 3 $\mathcal{L}\{tf(t)\} = -\dfrac{d}{ds}\mathcal{L}\{f(t)\}$.

Pf
$$\frac{d}{ds}\mathcal{L}\{f(t)\} = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{d}{ds}\,e^{-st}f(t)\,dt$$
$$= \int_0^\infty (-t)e^{-st}f(t)\,dt = \int_0^\infty e^{-st}(-t)f(t)\,dt = \mathcal{L}\{-tf(t)\}.$$

Ex 5 $\mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$, $\quad \mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$, $\quad$ when $s > a$.

Ex 6 $\mathcal{L}\{te^{at}\} = \dfrac{1}{(s-a)^2}$, when $s > a$.

Sol By Th 3 $\mathcal{L}\{te^{at}\} = -\dfrac{d}{ds}\,\dfrac{1}{s-a} = \dfrac{1}{(s-a)^2}$.

Ex 7 Find the solution of $y'' + y = \sin(2t)$, with initial data $y(0) = 2$, $y'(0) = 1$.
Sol Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the equation using the formulas in Th 1-2 we get:

$$\begin{aligned}\mathcal{L}\{y''(t) + y(t)\} &= \mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = s^2Y(s) - sy(0) - y'(0) + Y(s) \\ &= (s^2+1)Y(s) - 2s - 1 = \mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}\end{aligned}$$

Hence

$$Y(s) = \frac{1+2s}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

The inverse Laplace transform of the first part is by previous examples $\sin t + 2\cos t$. For the other part we put up partial fractions:

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

where A, B, C, D are to be determined. If we put the term on a common denominator again we get

$$\begin{aligned}&\frac{(As+B)(s^2+4) + (Cs+D)(s^2+1)}{(s^2+1)(s^2+4)} \\ &\qquad = \frac{(A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B + D}{(s^2+1)(s^2+4)} = \frac{2}{(s^2+1)(s^2+4)}\end{aligned}$$

Hence $A + C = B + D = 4A + C = 0$ and $4B + D = 2$ which gives $A = C = 0$ and $3B = 2$ so $B = 2/3$ and $D = -2/3$. Hence

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{2}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

so

$$Y(s) = 2\frac{s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}$$

Hence

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t)$$