Lecture 2: Section 1.2 Analytical solution of the simple models. We can actually solve the mice-owl model (1.1.3) analytically:

$$\frac{dp}{dt} = 0.5p - 450 \tag{1.2.1}$$

The general idea is that we will try to rewrite it as a an equation of the form

$$\frac{d}{dt}G(t, p(t)) = f(t), \tag{1.2.2}$$

that we can solve simply by integrating. If we rewrite our equation (1.2.1) as

$$\frac{dp/dt}{p - 900} = \frac{1}{2} \tag{1.2.3}$$

we see that by the chain rule for derivatives

$$\frac{d}{dt}G(p(t)) = G'(p(t))\frac{dp(t)}{dt}$$

we get

$$\frac{d}{dt}\ln|p - 900| = \frac{1}{2} \tag{1.2.4}$$

which is of the form (1.2.2). Integrating (1.2.4) gives

$$\ln|p - 900| = \frac{t}{2} + C$$

or after integration:

$$|p - 900| = e^C e^{t/2}$$

i.e.

$$p - 900 = \pm e^C e^{t/2} = ce^{t/2}$$

where $c = \pm e^C$, i.e.

$$p = 900 + ce^{t/2}$$

The constant c will be determined by initial conditions: If $p(0) = 850$, then $-50 = p - 900 = ce^0$, so $p = 950 - 50e^{t/2}$ which is indeed decreasing fast towards 0 Similarly if $p(0) = 950$, then $50 = p - 900 = ce^0$ so $p = 900 + 50e^{t/2}$, which is indeed increasing fast towards infinity. A similar calculation for the first model gives

$$v = 49 + ce^{-t/5}$$

from which we also see that as $t \to \infty$ any initial state tend to the equilibrium $v = 49$. As we shall see in section 2.1, this calculation gives that the **general solutions** to

$$\frac{dy}{dt} = ay - b, \tag{1.2.1}$$

is given by

$$y = b/a + ce^{at},$$

where the constant c is determined by satisfying an **initial condition** $y(0) = y_0$

1.3 Types of differential equations. An **ordinary differential equation** is a an equation with derivatives of the unknown with respect one variable only, e.g. the time time. We have seen two examples of this in the previous sections:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \qquad \frac{dp}{dt} = 0.5p - 450 \tag{1.3.1}$$

Another example is the equation for the displacement $u(t)$ from equilibrium of weight hanging in a spring under the influence of gravity:

$$mu''(t) + ku(t) = 0. \tag{1.3.2}$$

where k is the spring constant. The **order** of a differential equation is the highest order of derivatives in the equation, so e.g. the order of (1.3.1) is one and the order of (1.3.2) is two. A general n th order differential equation is of the form

$$F\big(t, u(t), u'(t), \dots, u^{(n)}(t)\big) = 0.$$

A special case are the so called **linear equations**

$$a_0(t)u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \cdots + a_n(t)u(t) = g(t).$$

The equations (1.3.1)-(1.3.3) are **linear equations**. A **nonlinear equation** for population growth is given by the logistic equation

$$\frac{dp}{dt} = (r - ap)p \tag{1.3.5}$$

taking into account that the food supply is limited. Another nonlinear equation is that of the displacement angle θ of an oscillating pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \tag{1.3.6}$$

Note however that for small displacement angles we can approximate $\sin\theta \sim \theta$ in which case we get a linear equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \tag{1.3.7}$$

This is called **linearization**.

A **partial differential equation** is an equation containing partial derivatives with respect to more than one variable, e.g. the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{1.3.3}$$

describing the displacement $u(t,x)$ of a guitar string from equilibrium along its length $0 \le x \le L$.

We further consider **systems of differential equations**, e.g. the Lotka-Volterra, predator-prey equation

$$\begin{aligned} dx/dt &= ax - \alpha xy, \\ dy/dt &= -cy + \gamma xy \end{aligned} \tag{1.3.4}$$

where $x(t)$ and $y(t)$ describes the respective populations of prey and predator.