Lecture 20: 7.1 Systems of first order differential equations. A second order equation can always be written as a first order system:

\[ \begin{align*}
  x'_1 &= x_2 \\
  x'_2 &= -kx_1
\end{align*} \]

First order systems also show up naturally not coming from a higher order equation:

Consider two interconnected tanks that contain water with a certain amount of salt \( Q_1 \) respectively \( Q_2 \) oz of salt. Suppose tank one contain 60 gal of water and tank two 100 gal. Suppose the water containing \( q_1 \) oz of salt per gal flows in to tank one at a rate of 3 gallons per min and \( q_2 \) oz of salt per gal flows in to tank two at a rate of 1 gallons per min. Suppose also that 4 gal per min flows out of tank one half of which flows in to tank two while the remainder leaves the system and 3 gal per min flows out of tank two, of which 1 gallon flows into tank one, and the rest leaves the system. The system of equations describing this is

\[ \begin{align*}
  Q'_1 &= 3q_1 + Q_2/100 - 4Q_1/60 \\
  Q'_2 &= q_2 + 2Q_1/60 - 3Q_2/100
\end{align*} \]

One could attempt to rewrite this as a second order equation for one unknown only \( Q = aQ_1 + bQ_2 \), but instead we will learn methods to directly solve systems.

A general first order \( 2 \times 2 \) system of differential equations can be written

\[ \begin{align*}
  x'_1 &= F_1(t, x_1, x_2) \\
  x'_2 &= F_2(t, x_1, x_2)
\end{align*} ; \quad x'_1(t_0) = x_1^0 \\
&\quad x'_2(t_0) = x_2^0
\]

or if we introduce vector notation

\[
\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \quad \begin{pmatrix} x' \\ F(x) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} , \quad x_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}
\]

we can write this in a more concise form:

\[ \begin{pmatrix} x' \\ F(t, x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \quad x(t_0) = x_0. \]

The same methods uses for one equations works to show that we have existence for the \( 2 \times 2 \) system. Using Euler’s method:

\[ \begin{align*}
  x_{n+1} &= x_n + F(t_n, x_n)(t_{n+1} - t_n), \\
  t_{n+1} &= t_0 + nh, \quad n \geq 0,
\end{align*} \]

gives and approximation for \( x(t_n) \approx x_n \). Alternatively, as before we can also prove existence with successive approximation

\[ \begin{align*}
  x_0(t) &= x_0, \\
  x_{n+1}(t) &= x_0 + \int_{t_0}^t F(s, x_n(s)) \, ds, \quad n \geq 0.
\end{align*} \]
7.2 (2x2) Linear systems with constant coefficients. We first consider a homogeneous 2 × 2 constant coefficient linear system of differential equations:

\begin{align*}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2
\end{align*}

(7.1.1)

Let us first consider a 2 × 2 linear system of algebraic equations

\begin{align*}
a_{11}x_1 + a_{12}x_2 &= y_1 \\
a_{21}x_1 + a_{22}x_2 &= y_2
\end{align*}

(7.1.2)

We will write this system in matrix form. Let \( A \) be the 2 × 2 matrix

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \]

i.e. a collection of 2 × 2 entries \( A = (a_{ij}), i, j = 1, 2 \), and let \( x \) be the 2 vector

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \]

We define the product of the 2 × 2 matrix \( A \) by the 2 vector \( x \) to be the 2 vector

\begin{align*}
Ax &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{21}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2
\end{align*}

(7.1.3)

i.e. the vector whose first component is the dot product \((a_{11}, a_{12}) \cdot (x_1, x_2) = a_{11}x_1 + a_{12}x_2\) of the first row of \( A \) and \( x \) and whose second component is the dot product \((a_{21}, a_{22}) \cdot (x_1, x_2) = a_{11}x_1 + a_{12}x_2\) of the second row of \( A \) and \( x \). As indicated above; another way to see this matrix product is as a linear combination of the column vectors: \( Ax = [a_1 \ a_2]x = a_1x_1 + a_2x_2 \).

If

\[ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \]

the algebraic system (7.1.2) can then be written

\[ Ax = y \]

and the system of differential equations (7.1.1) can be written

\[ \dot{x} = Ax \]

Any 2 × 2 matrix \( A \) determines a linear map

\[ \mathbb{R}^2 \ni x \to Ax \in \mathbb{R}^2 \]

Conversely, every linear map is given by matrix multiplication. If \( B \) is another 2 × 2 matrix

\[ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \]
then multiplication first by $B$ and then $A$

$$x \xrightarrow{\text{multiply by } B} Bx \xrightarrow{\text{multiply by } A} A(Bx)$$

defines a linear map $\mathbb{R}^2 \ni x \to A(Bx) \in \mathbb{R}^2$. This linear map corresponds to multiplying by some matrix. The matrix product $AB$ is constructed so that multiplying by the matrix $AB$

$$x \xrightarrow{\text{multiply by } AB} (AB)x$$
is the same as first multiplying by $B$ and then by $A$, i.e. $(AB)x = A(Bx)$.

If $A$ and $B$ are $2 \times 2$ matrices then the product $AB$ is the $2 \times 2$ matrix

$$(7.1.4) \quad AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$
i.e. the entry in the $i$th row and $j$th column of $AB$ is the dot product between the $i$th row of $A$ and the $j$th column of $B$: $a_{ij}b_{1j} + a_{ij}b_{2j}$. One can see the matrix product in terms of the column picture, if $\mathbf{B} = [\mathbf{b}_1 \mathbf{b}_2]$ then $AB = [\mathbf{Ab}_1 \mathbf{Ab}_2]$, i.e. if $\mathbf{b}_1, \mathbf{b}_2$ are the column vectors of $\mathbf{B}$ then $\mathbf{Ab}_1, \mathbf{Ab}_2$ are the column vectors of $AB$.

Ex 1

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \text{ rotates vectors an angle } \pi/2 \text{ counterclockwise.}$$

Ex 2

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix} \text{ scales vectors by a factor } 3.$$  

Ex 3

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ 3x_1 \end{bmatrix} \text{ scales and rotates.}$$

These maps are all invertible. However a projection is not:

Ex 4

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

Usually $AB \neq BA$. The $2 \times 2$ identity matrix $I$ is given by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying with it is similar to multiplying by 1:

$$Ix = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$$

An $2 \times 2$ matrix is called **invertible** if there is an $n \times n$ matrix $A^{-1}$ such that

$$AA^{-1} = A^{-1}A = I$$

It turns out that the algebraic system (7.1.2) can be solved only if the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$
is nonvanishing. In that case $A$ has an inverse given by

$$(7.1.5) \quad A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} \frac{a_{22}}{\det A} & -\frac{a_{12}}{\det A} \\ -\frac{a_{21}}{\det A} & \frac{a_{11}}{\det A} \end{bmatrix}$$

In fact by (7.1.4)

$$A^{-1}A = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \frac{1}{\det A} \begin{bmatrix} a_{22}a_{11} - a_{12}a_{21} & 0 \\ 0 & a_{21}a_{12} + a_{11}a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$
Furthermore, if \( \det A \neq 0 \) then the linear algebraic system has a unique solution:

\[ Ax = b \iff A^{-1}Ax = A^{-1}b \iff x = Ix = A^{-1}b \]

If \( \det A \neq 0 \) then the homogeneous problem \( b = 0 \), only has the trivial solution \( x = A^{-1}0 = 0 \). But if \( \det A = 0 \), then the homogeneous problem has a nontrivial solution \( x \neq 0 \) and the inhomogeneous problem, \( b \neq 0 \), might not have any solution.

**Ex 1** Solve the system

\[
\begin{align*}
x_1 + 3x_2 &= 5 \\
2x_2 + 4x_2 &= 6
\end{align*}
\]

**Sol** The system can be written in matrix form

\[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
6
\end{bmatrix}
\]

By (7.1.5)

\[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix}^{-1}
= 
\frac{1}{1 \cdot 4 - 3 \cdot 2}
\begin{bmatrix}
4 & -3 \\
-2 & 1
\end{bmatrix}
= 
\begin{bmatrix}
-2 & 3/2 \\
2 & -1/2
\end{bmatrix}
\]

Hence

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
-2 & 3/2 \\
1 & -1/2
\end{bmatrix}
\begin{bmatrix}
5 \\
6
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
2
\end{bmatrix}
\]

**Ex 2** Solve the system

\[
\begin{align*}
x_1 + 2x_2 &= 5 \\
2x_1 + 4x_2 &= 6
\end{align*}
\]

**Sol** Since the determinant of the system vanishes

\[
\begin{vmatrix}
1 & 2 \\
2 & 4
\end{vmatrix}
= 1 \cdot 4 - 2 \cdot 2 = 0
\]

the inverse does not exist. Subtracting twice the first equation from the second gives the equivalent system:

\[
\begin{align*}
x_1 + 2x_2 &= 5 \\
0 &= -4
\end{align*}
\]

The second equation can not hold so the system has no solutions.

**Ex 3** Solve the system

\[
\begin{align*}
x_1 + 2x_2 &= 0 \\
2x_1 + 4x_2 &= 0
\end{align*}
\]

**Sol** Subtracting twice the first equation from the second gives the equivalent system:

\[
\begin{align*}
x_1 + 2x_2 &= 0 \\
0 &= 0
\end{align*}
\]

The second equation always hold and the first equations has a whole line of solutions, if we put \( x_2 \) equal to a parameter \( \alpha \) we get

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
-2\alpha \\
\alpha
\end{bmatrix}
= \alpha
\begin{bmatrix}
-2 \\
1
\end{bmatrix}, \quad \text{for any } \alpha
\]