Lecture 21 7.2-3 Algebraic Systems.
Ex. 1 We want to solve the 3×3 system

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Geometrically this represents the intersection of 3 planes. To minimize the writing it is convenient to only write out the **coefficient matrix** and the column vector or to combine them in one to the **augmented matrix** of the system.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

We want to eliminate x_1 from the last equation by using the first:

$$\begin{array}{c} \text{[equation 3]} \\ +4\,\text{[equation 1]} \\ \hline \text{[new equation 3]} \end{array} \qquad \begin{array}{r} -4x_1 + 5x_2 + 9x_3 = -9 \\ 4x_1 - 8x_2 + 4x_3 = 0 \\ \hline -3x_2 + 13x_3 = -9 \end{array}$$

After some practice this calculation is usually performed mentally.
Hence we get the system (written in both ways for comparison)

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \ (3) + 4(1)$$

Now first multiply the second equation by $1/2$:

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \ (2)/2$$

We now want to eliminate x_2 from the last equation by adding 3 times the second:

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \ (3) + 3(2)$$

Hence we got an equivalent system in non-degenerate triangular form.
Because the diagonal entries are nonvanishing we can solve it using back substitution:

$$\begin{aligned} x_1 - 2x_2 \quad &= -3 \\ x_2 \quad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \ \begin{array}{l} (1) - (3) \\ (2) + 4(3) \\ \ \end{array}$$

Now having cleared up the column above x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Adding 2 times the second equation to the first gives

$$\begin{aligned} x_1 \quad\quad &= 29 \\ x_2 \quad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \ \begin{array}{l} (1) + 2(2) \\ \ \\ \ \end{array}$$

Ex 2 We can invert the matrix in the previous example

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

with the same method we used to solve the system in the previous example. The matrix A defines a map $\mathbf{R}^3 \ni \mathbf{x} \to \mathbf{y} = A\mathbf{x} \in \mathbf{R}^3$. The inverse matrix defines the inverse map $\mathbf{R}^3 \ni \mathbf{y} \to \mathbf{x} = A^{-1}\,\mathbf{y} \in \mathbf{R}^3$ taking $\mathbf{y}$ back to $\mathbf{x}$. Hence if we can solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= y_1 \\ 2x_2 - 8x_3 &= \qquad y_2 \\ -4x_1 + 5x_2 + 9x_3 &= \qquad\qquad y_3 \end{aligned}$$

for $\mathbf{x}$ as a linear function of $\mathbf{y}$ we would have found the inverse. We can solve this system the same way we solved the previous system. In order to keep track of the coefficient in front of the different components of $\mathbf{y}$ as well perform row operations on the larger augmented system

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -8 & 0 & 1 & 0 \\ -4 & 5 & 9 & 0 & 0 & 1 \end{array}\right]$$

As before we get

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -8 & 0 & 1 & 0 \\ 0 & -3 & 13 & -4 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ (3)+4(1) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1/2 & 0 \\ 0 & -3 & 13 & -4 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (2)/2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -4 & 3/2 & 1 \end{array}\right] \begin{array}{l} \\ \\ (3)+3(2) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 5 & -3/2 & 0 \\ 0 & 1 & 0 & -16 & 13/2 & 4 \\ 0 & 0 & 1 & -4 & 3/2 & 1 \end{array}\right] \begin{array}{l} (1)-(3) \\ (2)+4(3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -27 & 23/2 & 8 \\ 0 & 1 & 0 & -16 & 13/2 & 4 \\ 0 & 0 & 1 & -4 & 3/2 & 1 \end{array}\right] \begin{array}{l} (1)+2(2) \\ \\ \\ \end{array}$$

Writing this out as a system we have

$$\begin{aligned} x_1 \qquad\qquad &= -27y_1 + \tfrac{23}{2}y_2 + 8y_3 \\ x_2 \qquad &= -16y_1 + \tfrac{13}{2}y_2 + 4y_2 \\ x_3 &= -4y_1 + \tfrac{3}{2}y_2 + y_3 \end{aligned}$$

Hence

$$A^{-1} = \begin{bmatrix} -27 & 23/2 & 8 \\ -16 & 13/2 & 4 \\ -4 & 3/2 & 1 \end{bmatrix}$$

Ex 3 We want to solve the 3×3 system

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 2x_1 - 3x_2 - 2x_3 &= 2 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 2 & -3 & -2 & 2 \end{array}\right]$$

and hence

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 1 & -4 & 2 \end{array}\right] \ (3) - 2(1)$$

and

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 0 & -2 \end{array}\right] \ (3) - (2)/2$$

and writing out the system

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 0 &= -2 \end{aligned}$$

Here the last equation can not be satisfied which means there are no solutions.
Ex 4 However, if we instead consider

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 0 \\ 2x_1 - 3x_2 - 2x_3 &= 0 \end{aligned}$$

we get the system

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

which has infinitely many solutions, for any α;

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3\alpha \\ 4\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \alpha$$

Def A set of vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ is called **linearly independent** if the only numbers $c_1, \cdots, c_k$ such that

(7.1.6) $$c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_k = \mathbf{0}$$

holds are $c_1 = \cdots = c_k = 0$. The set is called **linearly dependent** if there are $c_1, ..., c_k$ not all zero such that (7.1.6) hold. If $k = 2$ the condition just says that one is not a multiple of the other. n vectors in $\mathbf{R}^n$ are linearly dependent if and only if the determinant of the matrix with those as it columns or rows vanish.

If A is the matrix in **Ex 4** then the column vectors of A are linearly dependent, for

$$A\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has the nonvanishing solution in **Ex 4**.