Lecture 22 7.3 Eigenvectors. The equation

$$(7.3.1) \qquad A\mathbf{x} = \mathbf{y} \qquad \text{or} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 &= y_1 \\ a_{21}x_1 + a_{22}x_2 &= y_2 \end{aligned}$$

can be viewed as a map transforming the vector $\mathbf{x} \in \mathbf{R}^2$ into the vector $\mathbf{y} \in \mathbf{R}^2$. Examples of such transformations are scalar multiplication or rotations of a vector. Vectors that are transformed into a multiple of themselves play an important role:

Ex 1 Let $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$, $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then $A\mathbf{x}^{(1)} = -\mathbf{x}^{(1)}$, $A\mathbf{x}^{(2)} = 3\mathbf{x}^{(2)}$.

Note that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a basis; any vector in the plane can be written

$$(7.3.2) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{x_1 + x_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x_1 - x_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}.$$

Knowing how A transforms $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ determines how it transforms any vector

$$A\mathbf{x} = A\big(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}\big) = c_1A\mathbf{x}^{(1)} + c_2A\mathbf{x}^{(2)} = -c_1\mathbf{x}^{(1)} + 3c_2\mathbf{x}^{(2)}$$

Def Let A be an $n \times n$ matrix. A scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$ is called an **eigenvalue** and the corresponding vector $\mathbf{x}$ is called an **eigenvector**.

How do we find eigenvectors? Well $A\mathbf{x} = \lambda\mathbf{x}$, for some $\mathbf{x} \neq 0$ is the same as

$$(7.3.3) \qquad (A - \lambda I)\mathbf{x} = 0, \qquad \mathbf{x} \neq 0$$

which is equivalent to that $A - \lambda I$ is not invertible which is equivalent to that

$$(7.3.4) \qquad p(\lambda) = \det(A - \lambda I) = 0$$

This is called the **characteristic polynomial** for the matrix A and is equal to

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{22} = 0$$

Ex 2 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.
Sol The eigenvalues are solution of the characteristic equation (7.3.4):

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -2 \\ -2 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 2^2 = (1 - \lambda - 2)(1 - \lambda + 2) = 0$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. The eigenvectors are solutions to (7.3.3): If $\lambda = \lambda_1 = -1$ then (7.3.4) becomes

$$(A - \lambda_1 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{aligned} 2\xi_1 - 2\xi_2 &= 0 \\ -2\xi_1 + 2\xi_2 &= 0 \end{aligned} \Leftrightarrow \begin{aligned} \xi_1 &= \alpha \\ \xi_2 &= \alpha \end{aligned}; \quad \boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $\lambda = \lambda_2 = 3$ then (7.3.4) becomes

$$(A - \lambda_2 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{aligned} -2\xi_1 - 2\xi_2 &= 0 \\ -2\xi_1 - 2\xi_2 &= 0 \end{aligned} \Leftrightarrow \begin{aligned} \xi_1 &= \beta \\ \xi_2 &= \beta \end{aligned}; \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We can pick any numbers $\alpha \neq 0$ and $\beta \neq 0$, e.g. $\alpha = 1$ and $\beta = 1$.

Ex 3 Find the eigevales and eigenspaces of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Sol Characteristic polynomial $(2-\lambda)^2(4-\lambda)$.

$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and $(A-2I)\mathbf{x} = \mathbf{0}$ has augmented matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow x_1 - x_2 - x_3 = 0$

and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha+\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ spann the eigenspace.

$A - 4I = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, augmented matrix $\begin{bmatrix} -2 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Leftrightarrow \begin{cases} x_1 = 0, \\ x_2 - x_3 = 0 \end{cases}$ and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma \\ \gamma \end{bmatrix} = \gamma \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Ex 4 Find the eigevales and eigenspaces of $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

Sol The eigenvalues $\det(A - \lambda I) = (\lambda-2)^2(\lambda-4) = 0$.

Basis for $\lambda = 2$: $A - 2I = \begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $(A-2I)\mathbf{x} = \mathbf{0}$ has augmented matrix

$\begin{bmatrix} 0 & 4 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow x_2 = x_3 = 0$ and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Basis for $\lambda = 4$: After some work we get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \beta \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$.

There are not three linearly independent eigenvectors.

7.5 Linear Systems of differential equations. Let us now return to the system

$$\text{(7.5.1)} \qquad \mathbf{x}' = A\mathbf{x}, \qquad \text{or} \qquad \begin{array}{l} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \end{array}.$$

How will we find solutions to such systems? Recall that we found solution to $x' = ax$ where a is a constant by trying with $x = ce^{rt}$. Then $x' = rce^{rt} = rx = ax$ if $r = a$. In fact the solution to the simple system

$$\text{(7.5.2)} \qquad \begin{array}{l} x_1' = r_1 x_1 \\ x_2' = r_2 x_2 \end{array} \qquad \text{is} \qquad \begin{array}{l} x_1 = c_1 e^{r_1 t} \\ x_2 = c_2 e^{r_2 t} \end{array}$$

Let us therefore try with

$$\text{(7.5.3)} \qquad \mathbf{x} = \boldsymbol{\xi} e^{rt}$$

where r is a number and $\boldsymbol{\xi} = (\xi_1, \xi_2)^T$ is a constant vector to be determined. For this to be a solution we must have

$$\mathbf{x}' = r\boldsymbol{\xi} e^{rt} = A\boldsymbol{\xi} e^{rt} = A\mathbf{x}$$

or if we cancel the scalar factor e^{rt}:

$$\text{(7.5.4)} \qquad (A - rI)\boldsymbol{\xi} = \mathbf{0}, \qquad \boldsymbol{\xi} \neq 0$$

Therefore (7.5.3) is a solution to (7.5.1) if r is an eigenvalue and $\boldsymbol{\xi}$ an eigenvector. We have seen that r is an eigenvalue if it satisfies the characteristic equation. If the characteristic equation has two distinct real roots $r_1 \neq r_2$, then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent. Each eigenvector corresponds to one solution and the system decouples into two equations in the two different directions corresponding to the eigenvector. It basically becomes the system (7.5.2) if we change coordinates so the coordinate axis are in the direction of the eigenvectors.

Ex 6 Find the solution to the system $\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \mathbf{x}$, with $\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$.

Sol In a previous ex. we showed that $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors with eigenvalues $r_1 = -1$ and $r_2 = 3$. It follows that for any constants c_1 and c_2

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + c_2 \boldsymbol{\xi}^{(2)} e^{r_2 t}$$

is a solution, and since $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$ form a basis we can find c_1 and c_2 such that

$$\mathbf{x}(0) = c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)}$$

In fact by (7.3.2)

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a+b}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{a-b}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence $c_1 = (a+b)/2$ and $c_2 = (a-b)/2$ so the solution is

$$\mathbf{x} = \frac{a+b}{2} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{a-b}{2} e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$