Lecture 23: 7.5 Linear systems of differential equations.

Ex 2 Find the solution to the system

$$\mathbf{x}' = A\mathbf{x}, \qquad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -6 & -2 \\ -2 & -9 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix},$$

Sol First we want to find the eigenvalues r and eigenvectors $\boldsymbol{\xi} \neq 0$:

$$(7.5.5) \qquad A\boldsymbol{\xi} = r\boldsymbol{\xi} \qquad \Leftrightarrow \qquad (A - rI)\boldsymbol{\xi} = \mathbf{0}.$$

The eigenvalues are solution of the characteristic equation:

$$0 = \det(A - rI) = \begin{vmatrix} -6-r & -2 \\ -2 & -9-r \end{vmatrix} = (-6-r)(-9-r) - 2^2 = r^2 + 15r + 50$$

$$= r^2 + 2\frac{15}{2}r + \Big(\frac{15}{2}\Big)^2 - \frac{225}{4} + \frac{200}{4} = \Big(r + \frac{15}{2}\Big)^2 - \Big(\frac{5}{2}\Big)^2 = \Big(r + \frac{15}{2} + \frac{5}{2}\Big)\Big(r + \frac{15}{2} - \frac{5}{2}\Big)$$

Hence $\det(A - rI) = (r+5)(r+10)$ so the eigenvalues are $r_1 = -5$ and $r_2 = -10$.
If $r = r_1 = -10$ then (7.5.5) becomes

$$(A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} 4\xi_1 - 2\xi_2 = 0 \\ -2\xi_1 + \xi_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \xi_1 = \alpha \\ \xi_2 = 2\alpha \end{matrix}; \quad \boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If $r = r_2 = -5$ then (7.5.5) becomes

$$(A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} -\xi_1 - 2\xi_2 = 0 \\ -2\xi_1 - 4\xi_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \xi_1 = 2\beta \\ \xi_2 = -\beta \end{matrix}; \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

where we picked $\alpha = \beta = 1$. We have found eigenvalues and eigenvectors so that $A\boldsymbol{\xi}^{(1)} = r_1\boldsymbol{\xi}^{(1)}$ and $A\boldsymbol{\xi}^{(2)} = r_2\boldsymbol{\xi}^{(2)}$. It follows that for any constants c_1 and c_2

$$\mathbf{x} = c_1 e^{r_1 t}\boldsymbol{\xi}^{(1)} + c_2 e^{r_2 t}\boldsymbol{\xi}^{(2)}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$. In fact, then

$$\mathbf{x}' = r_1 c_1 e^{r_1 t}\boldsymbol{\xi}^{(1)} + r_2 c_2 e^{r_2 t}\boldsymbol{\xi}^{(2)}$$

and

$$A\mathbf{x} = c_1 e^{r_1 t} A\boldsymbol{\xi}^{(1)} + c_2 e^{r_1 t} A\boldsymbol{\xi}^{(2)} = c_1 e^{r_1 t} r_1 \boldsymbol{\xi}^{(1)} + c_2 e^{r_1 t} r_2 \boldsymbol{\xi}^{(2)}.$$

Since $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$ are not parallel they form a basis and we can find c_1 and c_2 so that

$$\mathbf{x}(0) = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}$$

In fact

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} c_1 + 2c_2 = a \\ 2c_1 - c_2 = b \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} c_1 = (a+2b)/5 \\ c_2 = (2a-b)/5 \end{matrix}$$

and hence

$$(7.5.6) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{a+2b}{5}\, e^{-5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2a-b}{5}\, e^{-10t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Note that the solution (7.5.6) tend to $\mathbf{0}$ as $t \to \infty$ for any initial condition, i.e. $\mathbf{0}$ is a **stable equilibrium**. However, to conclude this it would have been sufficient to calculate the eigenvalues and note that both are negative.

The **Direction field** and **phase portrait** are pictures in the x_1x_2-plane. By evaluating and plotting the vector $A\mathbf{x}$ starting at a number of points $\mathbf{x}$ we get the direction field and the phase portrait is obtained by also drawing a few solution curves which are tangential to the direction fields. In particular if we do this in the above example we will see that all solution curves go towards the origin $\mathbf{0}$.

Discrete Dynamical systems, linear transformations and eigenvectors.
Ex 1 In a certain town, 30% of the married men get divorced each year and 20% of the single men get married each year. Suppose that initially there are 8000 married men and 2000 single men. What is the proportion of married as $k \to \infty$?
Sol Let

$$\mathbf{w}_k = \begin{bmatrix} w_{k1} \\ w_{k2} \end{bmatrix} = \begin{bmatrix} \text{number of married men after } k \text{ years} \\ \text{number of single men after } k \text{ years} \end{bmatrix}.$$

Let A be the 2×2 matrix such that

$$\mathbf{w}_{k+1} = A\mathbf{w}_k,$$

$$A = \begin{bmatrix} \text{proportion of married} & \text{proportion of single} \\ \text{that stays married in a year} & \text{that gets married in a year} \\ \text{proportion of married} & \text{proportion of single} \\ \text{that gets divorced in a year} & \text{that stays single in a year} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

$\mathbf{w}_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix}$. After the first year we get $\mathbf{w}_1 = A\mathbf{w}_0 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$.
After the second year we get $\mathbf{w}_2 = A\mathbf{w}_1 = A^2\mathbf{w}_0$ and so on:

$$\mathbf{w}_k = A^k \mathbf{w}_0$$

It seems like as $k \to \infty$, $\mathbf{w}_k$ converges: $\mathbf{w}_{10} = \begin{bmatrix} 4004 \\ 5996 \end{bmatrix}$, $\mathbf{w}_{20} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$, $\mathbf{w}_{30} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$.
In fact, any initial condition will converge to the **steady state** $(4000, 6000)^T$, for which the number of divorces $0.3 \cdot 4000$ is equal to the number of marriages $0.2 \cdot 6000$.
If we start with $\mathbf{x}_1 = (2, 3)^T$ proportional to the steady state we get back $\mathbf{x}_1$:

$$A\mathbf{x}_1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{x}_1$$

There is another vector $\mathbf{x}_2 = (-1, 1)^T$ that A acts on by simply multiplying by $1/2$:

$$A\mathbf{x}_2 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2$$

The vectors $\mathbf{x}_1$,$\mathbf{x}_2$ form a basis so we can write our initial condition in terms of these:

$$\mathbf{w}_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = 2000 \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 4000 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2000\mathbf{x}_1 - 4000\mathbf{x}_2$$

Then

$$\mathbf{w}_k = A^k\mathbf{w}_0 = 2000A^k\mathbf{x}_1 - 4000A^k\mathbf{x}_2 = 2000\mathbf{x}_1 - 4000\frac{1}{2^k}\mathbf{x}_2$$

and $\mathbf{w}_k \to 2000\mathbf{x}_1 = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$, as $k \to \infty$.
A scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$ is called an **eigenvalue** and a corresponding vector $\mathbf{x}$ is called an **eigenvector**.
We just calculated $A^k\mathbf{x}$ for large k using the eigenvalues and eigenvectors.
We express $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ in terms of the basis of eigenvectors $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $i = 1, 2$.
Change of coordinates $\mathbf{x} = P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where $P = [\,\mathbf{x}_1\ \mathbf{x}_2\,]$, so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}$. Then
$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{x}_1 + c_1\lambda_2^k\mathbf{x}_2 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PD^kP^{-1}\mathbf{x}$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.
Hence $A = PDP^{-1}$ and $A^k = \left(PDP^{-1}\right)^k = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^kP^{-1}$.