Lecture 24: 7.6 Complex eigenvalues.

Ex Find the solution to the system

$$\mathbf{x}' = A\mathbf{x}, \qquad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix},$$

Sol 1 First we want to find the eigenvalues r and eigenvectors $\boldsymbol{\xi} \neq 0$:

(7.6.1) $$A\boldsymbol{\xi} = r\boldsymbol{\xi} \qquad \Leftrightarrow \qquad (A - rI)\boldsymbol{\xi} = \mathbf{0}.$$

The eigenvalues are solution of the characteristic equation:

$$0 = \det(A - rI) = \begin{vmatrix} -1-r & -2 \\ 2 & -1-r \end{vmatrix} = (-1-r)^2 + 2^2 = (-1 - r - 2i)(-1 - r + 2i)$$

so the eigenvalues are $r_1 = -1 - 2i$ and $r_2 = -1 + 2i$, where $i = \sqrt{-1}$.

If $r = r_1 = 1 - 2i$ then (7.6.1) becomes

$$(A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} 2i\xi_1 - 2\xi_2 = 0 \\ 2\xi_1 + 2i\xi_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \xi_1 = \alpha \\ \xi_2 = \alpha i \end{matrix}; \quad \boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

If $r = r_2 = 1 + 2i$ then (7.6.1) becomes

$$(A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} -2i\xi_1 - 2\xi_2 = 0 \\ 2\xi_1 - 2i\xi_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \xi_1 = \beta \\ \xi_2 = -\beta i \end{matrix}; \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Then for any complex constants c_1 and c_2

$$\mathbf{x} = c_1 e^{r_1 t} \boldsymbol{\xi}^{(1)} + c_2 e^{r_2 t} \boldsymbol{\xi}^{(2)}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$. In fact, then

$$\mathbf{x}' = r_1 c_1 e^{r_1 t} \boldsymbol{\xi}^{(1)} + r_2 c_2 e^{r_2 t} \boldsymbol{\xi}^{(2)}$$

and

$$A\mathbf{x} = c_1 e^{r_1 t} A\boldsymbol{\xi}^{(1)} + c_2 e^{r_1 t} A\boldsymbol{\xi}^{(2)} = c_1 e^{r_1 t} r_1 \boldsymbol{\xi}^{(1)} + c_2 e^{r_1 t} r_2 \boldsymbol{\xi}^{(2)}.$$

Since $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$ are not parallel they form a basis and we can find c_1 and c_2 so that

$$\mathbf{x}(0) = c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)}$$

In fact

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} c_1 + c_2 = a \\ ic_1 - ic_2 = b \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} c_1 = (a - ib)/2 \\ c_2 = (a + ib)/2 \end{matrix}$$

and hence

(7.6.2) $$\mathbf{x} = \frac{a - ib}{2} e^{-t-2it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{a + ib}{2} e^{-t+2it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

This is real if a,b are real, as can be seen using Euler's formulas $e^{2it} = \cos 2t + i \sin 2t$.

Sol 2 Since A is real it follows that if r is an eigenvalue with eigenvector $\boldsymbol{\xi}$ then the complex conjugate of the eigenvalue $\overline{r}$ is also an eigenvalue with complex conjugate eigenvector $\overline{\boldsymbol{\xi}}$. In fact taking the complex conjugate of $A\boldsymbol{\xi} = r\boldsymbol{\xi}$ gives $A\overline{\boldsymbol{\xi}} = \overline{r}\,\overline{\boldsymbol{\xi}}$. Since $e^{rt}\boldsymbol{\xi}$ is a solution it follows that the complex conjugate $e^{\overline{r}t}\overline{\boldsymbol{\xi}}$ is a solution and so are the real and imaginary parts $\mathbf{u} = (e^{rt}\boldsymbol{\xi} + e^{\overline{r}t}\overline{\boldsymbol{\xi}})/2$ and $\mathbf{v} = (e^{rt}\boldsymbol{\xi} - e^{\overline{r}t}\overline{\boldsymbol{\xi}})/2i$. Writing $\boldsymbol{\xi} = \mathbf{a} + i\mathbf{b}$ and $r = \lambda + i\mu$ we get after some work two real solutions

$$\mathbf{u} = e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t),$$
$$\mathbf{v} = e^{\lambda t}(\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t)$$

and any solution can be written as

(7.6.3) $$\mathbf{x} = c_1 \mathbf{u} + c_2 \mathbf{v}.$$

Note that in (7.6.2) $\mathbf{x} \to \mathbf{0}$, as $t \to \infty$, so $\mathbf{0}$ is a stable equilibrium. In fact this follows since the real part of the eigenvalues are negative.