Lecture 25: 7.8 Repeated eigenvalues. Recall first that if A is a 2×2 matrix and the characteristic polynomial have two distinct roots $r_1 \neq r_2$ then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent for if $c_1\mathbf{x}^{(1)}+c_2\mathbf{x}^{(2)} = 0$ it follows that $\mathbf{0} = c_1 A\mathbf{x}^{(1)} + c_2 A\mathbf{x}^{(2)} = c_1 r_1 \mathbf{x}^{(1)} + c_2 r_2 \mathbf{x}^{(2)}$ which if we multiply the previous equation by r_1 and subtract them leads to $c_2(r_1 - r_2)\mathbf{x}^{(2)} = 0$ and hence $c_2 = 0$, and therefore $c_1 = 0$. Suppose now that we have a 2×2 matrix A with double eigenvalue $p\,(r) = (r - \lambda)^2$. We always have one eigenvector:

$$(A - \lambda I)\boldsymbol{\xi} = 0, \qquad \boldsymbol{\xi} \neq 0 \tag{7.8.1}$$

but do we have another eigenvector that is not just a multiple of $\boldsymbol{\xi}$? If $A{=}\lambda I$ then any vector is an eigenvector, e.g. $(1,0)^T$, $(0,1)^T$. But if $A{\neq}\lambda I$ then there is no other nonparallel eigenvector. In fact, if there was another nonparallel eigenvector $\boldsymbol{\zeta}$ then $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ would form a basis so any vector can be written as a linear combination of them and hence $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$ which implies that $A - \lambda I = 0$.

In case there is no other non parallel eigenvector we claim that we instead can find a so called **generalized eigenvector** η such that

$$(A - \lambda I)\boldsymbol{\eta} = \boldsymbol{\xi} \tag{7.8.2}$$

It is not obvious that this can be solved since $A - \lambda I$ is not invertible. In fact the image of $A - \lambda I$ is a line (if it wasn't we could of course solve it for any right hand side). To prove it we pick any vector η such that $\boldsymbol{\zeta} = (A - \lambda I)\boldsymbol{\eta} \neq 0$. Then $(A - \lambda I)\boldsymbol{\zeta} = \mu\boldsymbol{\zeta}$ for some $|mu$, since the image is a line in the direction of $\boldsymbol{\zeta}$. But then $\boldsymbol{\zeta}$ would be an eigenvector of A with eigenvalue $\lambda + \mu$, contradicting that λ was the only eigenvalue, unless $\mu = 0$ in which case $\boldsymbol{\zeta}$ is a multiple of $\boldsymbol{\xi}$ in which case (7.8.2) can be solved. This alternatively follows from

Cayley Hamilton's theorem An $n \times n$ matrix A satisfies its own characteristic equation, i.e. if $p_A(r) = \det(A - rI) = a_n r^n + ... + a_0 = (\lambda_n - r)\cdots(\lambda_n - r)$ then

$$p_A(A) = a_n A^n + \cdots + a_0 I = (\lambda_n I - A)\cdots(\lambda_1 I - A) = 0$$

In the case above it would simply say that $(A - \lambda I)^2 = 0$, i.e. for any $\boldsymbol{\eta}$, $(A - \lambda I)\boldsymbol{\eta} = c_1\boldsymbol{\xi}$ for some c_1, since $(A - \lambda I)(A - \lambda I)\boldsymbol{\eta} = 0$.

Ex 1 Find the vectors $\boldsymbol{\xi}$ and η such that (7.8.1)-(7.8.2) hold if

$$A = \begin{bmatrix} \lambda & d \\ 0 & \lambda \end{bmatrix}, \qquad \text{where} \qquad d \neq 0, \tag{7.8.3}$$

Sol First we note that $p\,(r){=}(r{-}\lambda)^2$. The only eigenvector is parallel to $\boldsymbol{\xi}{=}(1,0)^T$;

$$(A - \lambda I)\boldsymbol{\xi} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad d\,\xi_2 = 0 \quad \Leftrightarrow \quad \xi_2 = 0 \tag{7.8.4}$$

and e.g. $\boldsymbol{\eta} = (0, 1/d)^T$ satisfies (7.8.2) since

$$(A - \lambda I)\boldsymbol{\eta} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad d\,\eta_2 = 1 \quad \Leftrightarrow \quad \eta_2 = 1/d \tag{7.8.5}$$

Assume now that λ is a double eigenvalue for A and that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are such that (7.8.1) and (7.8.2) hold. Then one solution to

$$\mathbf{x}' = A\mathbf{x}$$

is

$$\mathbf{x}_1 = e^{\lambda t}\boldsymbol{\xi}.$$

We claim that

$$\mathbf{x}_2 = te^{\lambda t}\boldsymbol{\xi} + e^{\lambda t}\boldsymbol{\eta} \tag{7.8.6}$$

is another solution. In fact, then

$$\mathbf{x}_2' = e^{\lambda t}\boldsymbol{\xi} + t\lambda_1 e^{\lambda t}\boldsymbol{\xi} + \lambda e^{\lambda t}\boldsymbol{\eta}$$

and by (7.8.2)

$$A\mathbf{x}_2 = te^{\lambda t}A\boldsymbol{\xi} + e^{\lambda t}A\boldsymbol{\eta} = te^{\lambda t}\lambda\boldsymbol{\xi} + e^{\lambda t}(\lambda\boldsymbol{\eta} + \boldsymbol{\xi}).$$

The general solution is therefore

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 e^{\lambda t}\boldsymbol{\xi} + c_2\big(te^{\lambda t}\boldsymbol{\xi} + e^{\lambda t}\boldsymbol{\eta}\big).$$

Ex 2 Find the solution to the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$, where $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.
Sol This is the matrix of the form (7.8.3) with $\lambda_1 = 2$ and $d = 3$ so by (7.8.4) $(1,0)^T$ is an eigenvector so one solution is $e^{2t}(1,0)^T$ and another solution is given by (7.8.6), where by (7.8.5) $\boldsymbol{\eta} = (0, 1/3)^T$. Hence the general solution is

$$\mathbf{x} = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\left(te^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ 1/3 \end{bmatrix}\right),$$

and the initial condition is satisfied if

$$\mathbf{x}(0) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1/3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad \Leftrightarrow \qquad \begin{matrix} c_1 = a \\ c_2 = 3b \end{matrix}.$$

Sol 2 One can alternatively solve this system directly because its in triangular form.

Triangularization or Jordan normal form If the matrix is not in triangular form one can transform to triangular form.

Note that the matrix above satisfies

$$\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Other matrices that satisfy $B^2 = 0$ are e.g

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$