Lecture 26: 7.7 The exponential matrix.
We will now find a nice way to the express the solution to the system

$$\mathbf{x}' = A\mathbf{x}, \tag{7.7.1}$$

where A is a 2×2 matrix, analogous to the formula for the solution of one equation. We can find two solutions to (7.7.1); $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ satisfying the initial conditions

$$\mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{7.7.2}$$

Using these two solutions we can express any solution

$$\mathbf{x} = a\mathbf{x}_1 + b\mathbf{x}_2 \tag{7.7.3}$$

where the constants a and b are determined by the initial condition:

$$\mathbf{x}(0) = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{7.7.4}$$

Let $\Phi(t)$ be the 2×2 matrix with columns the 2 vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$:

$$\Phi = [\,\mathbf{x}_1\ \mathbf{x}_2\,] = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \qquad \text{where} \qquad \mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \tag{7.7.5}$$

Then in view of the definition of matrix multiplication, (7.7.3) can be written

$$\mathbf{x}(t) = [\,\mathbf{x}_1\ \mathbf{x}_2\,]\begin{bmatrix} a \\ b \end{bmatrix} = \Phi(t)\mathbf{x}(0) \tag{7.7.6}$$

Now, suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental solution set but do not satisfy the initial conditions (7.7.2). The we can still write any solution $|boldx$ in the form (7.7.3) although the constants a and b can not be as easily determined from the initial condition as in (7.7.4). However, if we form $\Psi = [\,\mathbf{x}_1 \quad \mathbf{x}_2\,]$ we can still write $\mathbf{x}(t) = \Psi(t)\,[\,a \quad b\,]$ but we must solve $\mathbf{x}(0) = \Psi(0)\,[\,a \quad b\,]$, i.e $[\,a \quad b\,] = \Psi(0)^{-1}\mathbf{x}(0)$ so in that case $\mathbf{x}(t) = \Psi(t)\Psi(0)^{-1}\mathbf{x}(0)$.
(7.7.6) means that when we calculated $\Phi(t)$ we can find any solution to (7.7.1) by just multiplying $\Phi(t)$ by the initial conditions $\mathbf{x}(0)$. Summarizing we have found:

Th 1 Given a 2×2 matrix A, there is a 2×2 matrix $\Phi(t)$ such that any solution of

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{7.7.7}$$

satisfies

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 \tag{7.7.8}$$

Note the analogy with the case of one equation

$$x' = ax, \qquad x(0) = x_0$$

where the solution satisfies

$$x(t) = e^{at}x_0.$$

The analogy actually goes further. Recall that we can expand in a Taylor series

$$e^{at} = 1 + ta + \frac{1}{2}t^2a^2 + \cdots + \frac{1}{k!}t^ka^k + \cdots$$

If A is a 2×2 matrix now define the 2×2 **exponential matrix** by

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots \tag{7.7.7}$$

Each term is a 2×2 matrix and one can show that each entry in the sum converges. It is not practical to use (7.7.7) but there are other ways to calculate it.
We will show that e^{At} is in fact equal to the matrix $\Phi(t)$ in Th 1. In fact,

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\Big(I + tA + \frac{t^2}{2}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots\Big)$$

$$= A + tA^2 + \cdots + \frac{t^{k-1}}{(k-1)!}A^k + \cdots = A\Big(I + tA + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1}\Big) = Ae^{At}$$

and if $t = 0$

$$e^{A0} = I$$

It therefore follows that

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

satisfies (7.7.5) and $e^{At} = \Phi(t)$ in (7.7.6). In fact,

$$\frac{d}{dt}\mathbf{x} = \frac{d}{dt}e^{At}\mathbf{x}_0 = \Big(\frac{d}{dt}e^{At}\Big)\mathbf{x}_0 = Ae^{At}\mathbf{x}_0 = A\mathbf{x},$$

and

$$\mathbf{x}(0) = e^{A0}\mathbf{x}_0 = I\mathbf{x}_0 = \mathbf{x}_0$$

Ex 1 Calculate the exponential matrix for the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

Sol By Ex 7.3.2 the eigenvalues and vectors are $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and by Ex 7.5.1 the solution to $\mathbf{x}' = A\mathbf{x}$ with any initial data $\mathbf{x}(0) = \mathbf{x}_0 = (a, b)^T$ is

$$\mathbf{x} = \frac{a+b}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{a-b}{2}e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-t} + e^{3t}}{2} \\ \dfrac{e^{-t} - e^{3t}}{2} \end{bmatrix} a + \begin{bmatrix} \dfrac{e^{-t} - e^{3t}}{2} \\ \dfrac{e^{-t} + e^{3t}}{2} \end{bmatrix} b$$

$$= \begin{bmatrix} \dfrac{e^{-t} + e^{3t}}{2} & \dfrac{e^{-t} - e^{3t}}{2} \\ \dfrac{e^{-t} - e^{3t}}{2} & \dfrac{e^{-t} + e^{3t}}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Hence the solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0, \quad \text{where} \quad \Phi(t) = \begin{bmatrix} \dfrac{e^{-t} + e^{3t}}{2} & \dfrac{e^{-t} - e^{3t}}{2} \\ \dfrac{e^{-t} - e^{3t}}{2} & \dfrac{e^{-t} + e^{3t}}{2} \end{bmatrix} = e^{At}.$$

Diagonalization. Since its so easy to calculate A^k applied to an eigenvector $A^k\mathbf{x}^{(n)} = \lambda_n^k\mathbf{x}^{(n)}$ and since we can expand any vector as a sum of the eigenvectors if they form a basis, there should be an easy way to calculate A^k in this case. In fact, note that in our example above

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

Hence if we multiply on the right withe the inverse we get

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = TDT^{-1}$$

where D is the diagonal matrix in the middle. It therefore follows that

$$A^k = TDT^{-1}TDT^{-1}\cdots TDT^{-1} = TD^kT^{-1}$$

Here

$$D^k = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \cdots \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} (-1)^k & 0 \\ 0 & 3^k \end{bmatrix}$$

Hence its easy to calculate

$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

Hence

$$e^{At} = I + TDT^{-1}t + TD^2T^{-1}\frac{t^2}{2} + \cdots = T(I + Dt + D^2\frac{t^2}{2} + \dots)T^{-1} = Te^{Dt}T^{-1}$$

since we also can write $I = TT^{-1}$. Here

$$\begin{aligned} e^{Dt} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} t + \begin{bmatrix} (-1)^2 & 0 \\ 0 & 3^2 \end{bmatrix} \frac{t^2}{2} + \dots \\ &= \begin{bmatrix} 1 - t + (-1)^2\frac{t^2}{2} + \dots & 0 \\ 0 & 1 + 3t + 3^2\frac{t^2}{2} + \dots \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \end{aligned}$$

Hence

$$e^{At} = Te^{Dt}T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \cdots = \begin{bmatrix} \dfrac{e^{-t} + e^{3t}}{2} & \dfrac{e^{-t} - e^{3t}}{2} \\ \dfrac{e^{-t} - e^{3t}}{2} & \dfrac{e^{-t} + e^{3t}}{2} \end{bmatrix}$$

which agrees with the previous answer.

Note that the differential equation $\mathbf{x}' = A\mathbf{x}$, where A is the matrix above can be solved by diagonalizing the whole system as follows: Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$\mathbf{y}' = T^{-1}\mathbf{x}' = T^{-1}AT\mathbf{y} = D\mathbf{y}$$

i.e.

$$y_1' = \lambda_1 y_1, \qquad y_2' = \lambda_2 y_2$$

which has the solution $y_1 = c_1 e^{\lambda_1 t}$, $y_2 = c_2 e^{\lambda_2 t}$ and we can transform back to $\mathbf{x} = T\mathbf{y}$.