Lecture 28: 9 Nonlinear Equations and stability. In this chapter we will consider autonomous 2×2 nonlinear systems

$$(9.1) \qquad \mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

Autonomous mean that $\mathbf{f}$ does not explicitly depend on t (but only through $\mathbf{x}$).

Def A point $\mathbf{x}^0$, where $\mathbf{f}(\mathbf{x}^0) = \mathbf{0}$ is called a **critical point**. If $\mathbf{x}^0$ is a critical point then $\mathbf{x}(t) \equiv \mathbf{x}^0$ is a solution to (9.1) called the **equilibrium solution**.

We want to study if the equilibrium solution is asymptotically stable, i.e. if solutions that start close to it will converge to it as time goes to infinity.
We will also introduce a weaker condition of stability, requiring that a solution that starts close to the equilibrium solution remains close to it.
Let us give the precise mathematical definitions of these concepts

Def A critical point $\mathbf{x}^0$ is said to be **stable** if, given any $\varepsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(0) - \mathbf{x}^0\| < \delta$ implies that $\|\mathbf{x}(t) - \mathbf{x}^0\| < \varepsilon$, for all $t > 0$. If its not stable its said to be **unstable**. A critical point $\mathbf{x}^0$ is said to be **asymptotically stable** if there is a $\delta > 0$ such that $\|\mathbf{x}(0) - \mathbf{x}^0\| < \delta$ implies that $\mathbf{x}(t) \to \mathbf{x}^0$, as $t \to \infty$.

A nonlinear example that well illustrates these concepts is that of the pendulum. The pendulum consists of a mass m attached to one end of a rigid weightless rod of length L. The other end of the rod is attached to a fixed origin O, around which the rod is free to rotate in a vertical plane. The position of the pendulum is described by an angle θ between the rod and the downward vertical direction, in which direction the gravitational force mg acts. Moreover we assume that we have a damping force or friction $-cL\theta'$ which is proportional to but in opposite direction to the velocity $L\theta'$. Newton's equation $ma = F$ gives

$$mL\frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mg\sin\theta$$

With the constants $\omega^2 = \frac{g}{L}$, $\gamma = \frac{c}{m}$ we write it as a system for $x_1 = \theta$ and $x_2 = \theta'$:

$$x_1' = x_2, \qquad x_2' = -\omega^2 \sin x_1 - \gamma x_2$$

The critical points are given by

$$x_2 = 0, \qquad -\omega^2 \sin x_1 - \gamma x_2 = 0,$$

i.e. $x_2 = 0$ and $x_1 = 0$ or $x_1 = \pi$. $x_1 = 0$ corresponds to the downward position. The downward position is stable. However its asymptotically stable only if the damping constant $c > 0$. $x_1 = \pi$ corresponds to the position of the rod straight up, which is an equilibrium if there is no initial velocity, since the forces in this case only acts straight down. However, this equilibrium is highly unstable and the smallest movement from it will lead to large movements of the rod.

Linear approximation to nonlinear systems. We want to investigate the behavior, in particular the stability, of a nonlinear system close to a critical point. We claim that in most cases there is a linear system whose trajectories qualitatively resemble those of the nonlinear system close to the critical point. First we note that we can always make a translation so that the critical point is the origin. In fact, if $\mathbf{x}^0$ is a critical point then we introduce a new variable $\mathbf{u} = \mathbf{x} - \mathbf{x}^0$ and we get that $\mathbf{u}' = \mathbf{x}' = \mathbf{f}(\mathbf{x}) = \mathbf{h}(\mathbf{u})$, where $\mathbf{h}(\mathbf{u}) = \mathbf{f}(\mathbf{u} + \mathbf{x}^0)$ has a critical point at $\mathbf{u} = \mathbf{0}$. Hence we may for this argument assume that the critical $\mathbf{x}^0 = \mathbf{0}$. If $\mathbf{f}$ is a differentiable function then by the linear approximation

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x} + \mathbf{g}(\mathbf{x}),$$

where

$$\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} \to 0, \qquad \text{as} \quad \mathbf{x} \to \mathbf{0}.$$

In fact

$$A = \mathbf{Df}(\mathbf{0}), \qquad \text{where} \quad \mathbf{Df} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$$

We further assume that the critical point is **isolated** which mean that there is no other critical point in a neighborhood of it. This implies that $\det A \neq 0$. In the above situation it is therefore reasonable to assume that the linear system

$$\mathbf{x}' = A\mathbf{x}$$

resembles the nonlinear system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

close to $\mathbf{x} \sim \mathbf{0}$ since $\mathbf{g}(\mathbf{x})$ is then small compared to $A\mathbf{x}$. Hence if the linear system is stable small perturbations of the zero solution remains small and hence we can expect the influence of $\mathbf{g}(\mathbf{x})$ to be small.