Lecture 3: 2.1 First order linear equations: Integrating Factor. In this lecture we will learn to solve a general first order linear differential equation

$$\frac{dy}{dt} + p(t)y = g(t). \tag{2.1.1}$$

We want to find a way to write this equation in the form

$$\frac{d}{dt}G(t, y(t)) = f(t), \tag{2.1.2}$$

because then we can integrate it and solve for y. There is actually a way to do by multiplying by a specially chosen integrating factor $\mu(t)$

$$\mu\frac{dy}{dt} + p(t)\mu y = \mu g(t),$$

and using the formula for the derivative of a product:

$$\frac{d}{dt}\big(\mu y\big) = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y.$$

In fact if we choose μ so that

$$\frac{d\mu}{dt} = p\mu \tag{2.1.3}$$

we see that

$$\frac{d}{dt}(\mu y) = \mu g$$

which is of the form (2.1.2) and we can integrate and solve for y. It remains to see if we can find $\mu(t)$ satisfying (2.1.3). Dividing by μ and using the chain rule we get

$$\frac{d}{dt}\ln|\,\mu| = \frac{d\mu/dt}{\mu} = p,$$

and integrating we get

$$\ln|\mu| = \int p\ dt + C$$

so

$$|\mu| = e^{\int p\ dt}e^{C}.$$

Note that we only need to find one solution μ so we can pick

$$\mu = e^{\int p\ dt}.$$

Hence

$$\frac{d}{dt}\left(e^{\int p\ dt}y\right) = e^{\int p\ dt}g$$

so integrating gives

$$e^{\int p\ dt}y = \int e^{\int p\ dt}g\,dt + C$$

and hence

$$y = Ce^{-\int p\ dt} + e^{-\int p\ dt}\int e^{\int p\ dt}g\,dt$$

Example Find the general solution to

$$\frac{dy}{dt} + ay = b \tag{2.1.4}$$

The integrating factor is

$$\mu = e^{\int a dt} = e^{at}.$$

If we multiply both sides of (2.1.4) with the integrating factor we get

$$e^{at}\frac{dy}{dt} + ae^{at}y = be^{at} \tag{2.1.4}$$

Hence we get

$$\frac{d}{dt}\Big(e^{at}y(t)\Big) = be^{at}$$

and if we take the antiderivative of this we get

$$e^{at}y(t) = \frac{b}{a}e^{at} + c$$

i.e.

$$y = b/a + ce^{-at}$$

Example Find all solutions to

$$\frac{dy}{dt} + \frac{1}{t}y = 1, t \qquad t > 0.$$

The integrating factor is

$$\mu = e^{\int t^{-1}\ dt} = e^{\ln|t|} = |\,t|$$

and multiplying with it gives

$$\frac{d}{dt}(ty) = t\Big(\frac{dy}{dt} + y\Big) = t$$

and integrating gives

$$ty = t^2/2 + C$$

so for some constant C

$$y = t/2 + C/t.$$

Example Find the general solution to

$$y' - 2y = 3e^t$$

Example Find the general solution to

$$y' - \frac{1}{2}y = 2\cos t$$