Lecture 3: 2.1 First order linear equations: Integrating Factor. In this lecture we will learn to solve a general first order linear differential equation

\[ \frac{dy}{dt} + p(t)y = g(t). \]

We want to find a way to write this equation in the form

\[ \frac{d}{dt} G(t, y(t)) = f(t), \]

because then we can integrate it and solve for \( y \). There is actually a way to do by multiplying by a specially chosen integrating factor \( \mu(t) \)

\[ \mu \frac{dy}{dt} + p(t) \mu y = \mu g(t), \]

and using the formula for the derivative of a product:

\[ \frac{d}{dt} (\mu y) = \mu \frac{dy}{dt} + \frac{d\mu}{dt} y. \]

In fact if we choose \( \mu \) so that

\[ \frac{d\mu}{dt} = p\mu \]

we see that

\[ \frac{d}{dt} (\mu y) = \mu g \]

which is of the form (2.1.2) and we can integrate and solve for \( y \). It remains to see if we can find \( \mu(t) \) satisfying (2.1.3). Dividing by \( \mu \) and using the chain rule we get

\[ \frac{d}{dt} \ln |\mu| = \frac{d\mu/dt}{\mu} = p, \]

and integrating we get

\[ \ln |\mu| = \int p \, dt + C \]

so

\[ |\mu| = e^{\int p \, dt} e^C. \]

Note that we only need to find one solution \( \mu \) so we can pick

\[ \mu = e^{\int p \, dt}. \]

Hence

\[ \frac{d}{dt} \left( e^{\int p \, dt} y \right) = e^{\int p \, dt} g \]

so integrating gives

\[ e^{\int p \, dt} y = \int e^{\int p \, dt} g \, dt + C \]

and hence

\[ y = Ce^{-\int p \, dt} + e^{-\int p \, dt} \int e^{\int p \, dt} g \, dt \]
**Example** Find the general solution to

\[
\frac{dy}{dt} + ay = b
\]

The integrating factor is

\[
\mu = e^{\int adt} = e^{at}.
\]

If we multiply both sides of (2.1.4) with the integrating factor we get

\[
e^{at}\frac{dy}{dt} + ae^{at}y = be^{at}
\]

Hence we get

\[
\frac{d}{dt} (e^{at}y(t)) = be^{at}
\]

and if we take the antiderivative of this we get

\[
e^{at}y(t) = \frac{b}{a}e^{at} + c
\]

i.e.

\[
y = \frac{b}{a} + ce^{-at}
\]

**Example** Find all solutions to

\[
\frac{dy}{dt} + \frac{1}{t}y = 1, \quad t > 0.
\]

The integrating factor is

\[
\mu = e^{\int \frac{1}{t} \, dt} = e^{\ln|t|} = |t|
\]

and multiplying with it gives

\[
\frac{d}{dt} (ty) = t \left( \frac{dy}{dt} + y \right) = t
\]

and integrating gives

\[
ty = t^2/2 + C
\]

so for some constant \( C \)

\[
y = t/2 + C/t.
\]

**Example** Find the general solution to

\[
y' - 2y = 3e^t
\]

**Example** Find the general solution to

\[
y' - \frac{1}{2}y = 2 \cos t
\]