Lecture 30: 9.2-3. We consider a 2×2 nonlinear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}. \tag{9.1}$$

We study the stability of the nonlinear system around critical points $\mathbf{f}(\mathbf{x}^0) = \mathbf{0}$ by approximating with a linear system as follows.

If $\mathbf{f}$ is twice continuously differentiable then by Taylor's formula

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^0) + (\mathbf{Df})(\mathbf{x}^0)(\mathbf{x} - \mathbf{x}^0) + \mathbf{h}(\mathbf{x}) \tag{9.2}$$

where the derivative is the linear map

$$(\mathbf{Df})(\mathbf{x}^0)(\mathbf{x} - \mathbf{x}^0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial f_1}{\partial x_2}(x_1^0, x_2^0) \\ \frac{\partial f_2}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial f_2}{\partial x_2}(x_1^0, x_2^0) \end{bmatrix} \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{bmatrix}$$

and

$$\mathbf{h}(\mathbf{x}) = \begin{bmatrix} h_1(x_1, x_2) \\ h_2(x_1, x_2) \end{bmatrix}$$

satisfies

$$\|\mathbf{h}(\mathbf{x})\| \le C\|\mathbf{x} - \mathbf{x}^0\|^2$$

In fact this follows from Taylor's formula applied to each component of $\mathbf{f}$:

$$f_1(x_1, x_2) = f_1(x_1^0, x_2^0) + \frac{\partial f_1}{\partial x_1}(x_1^0, x_2^0)(x_1 - x_1^0) + \frac{\partial f_1}{\partial x_2}(x_1^0, x_2^0)(x_2 - x_2^0) + h_1(x_1, x_2)$$
$$f_2(x_1, x_2) = f_2(x_1^0, x_2^0) + \frac{\partial f_2}{\partial x_1}(x_1^0, x_2^0)(x_1 - x_1^0) + \frac{\partial f_2}{\partial x_2}(x_1^0, x_2^0)(x_2 - x_2^0) + h_2(x_1, x_2)$$

where $|h_i(x_1, x_2)| \le C\big((x_1 - x_1^0)^2 + (x_2 - x_2^0)^2\big)$, $i = 1, 2$.

If $\mathbf{x}^0$ is a critical point and $\mathbf{x}$ is close to $\mathbf{x}^0$ then $\mathbf{u} = \mathbf{x} - \mathbf{x}^0$ is small and using (9.2) we rewrite the system (9.1) as

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{h}(\mathbf{x}^0 + \mathbf{u}), \qquad \text{where} \quad \|\mathbf{h}(\mathbf{x}^0 + \mathbf{u})\| \le C\|\mathbf{u}\|^2 \tag{9.3}$$

and $A = (\mathbf{Df})(\mathbf{x}_0)$. Assuming that A is invertible it follows that $\|\mathbf{h}(\mathbf{x}^0 + \mathbf{u})\| << \|A\mathbf{u}\|$ when $\|\mathbf{u}\|$ is small. We can hence approximate the nonlinear system (9.3) when $\mathbf{x}$ is close to $\mathbf{x}^0$ by the linear system

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

Ex Find the critical points and determine if they are stable

$$x_1' = -(x_1 - x_2)(1 - x_1 - x_2) \equiv f_1(x_1, x_2)$$
$$x_2' = x_1(2 + x_2) \equiv f_1(x_1, x_2) \tag{9.4}$$

The critical points are the solution to $f_1(x-1, x_2) = f_2(x_1, x_2) = 0$. There are four critical points (0.0), $(0, 1)$, $(-2, 2)$ and $(3, -2)$.

Around $(0, 0)$ we have

$$f_1(x_1, x_2) = -x_1 + x_2 + x_1^2 - x_2^2$$
$$f_2(x_1, x_2) = 2x_1 + x_1x_2$$

Hence

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1^2 - x_2^2 \\ x_1x_2 \end{bmatrix}$$

When $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ is small we can hence approximate the nonlinear system (9.4) by the linear system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The characteristic polynomial is $(-1-\lambda)(-\lambda) - 2 = \lambda^2 + \lambda - 2 = (\lambda + \frac{1}{2})^2 - \frac{9}{4} = 0$. The roots are $\lambda = -2, 1$. The eigenvector corresponding to $\lambda = -2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The linear system has a saddle point so its unstable.

Around $(-2, -2)$ we change variable $x_1 = u_2 - 2$, $x_2 = u_2 - 2$ and obtain

$$f_1(u_1 - 2, u_2 - 2) = (-u_1 + 2)(5 - u_1 - u_2) = -5u_1 + 5u_2 + u_1^2 - u_2^2$$
$$f_2(u_1 - 2, u_2 - 2) = (u_1 - 2)u_2 = -2u_2 + u_1u_2$$

Hence the nonlinear system cab be approximated by the linear system,

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The eigenvalues are -5 and -2. The linear system is hence stable.

We finally note that the matrices above can be obtain from the derivative matrix

$$\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} -1 + 2x_1 & 1 - 2x_2 \\ 2 + x_2 & x_1 \end{bmatrix}$$

Hence

$$(\mathbf{Df})(-2, -2) = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$$