Lecture 31: 9.2-3 Trajectories. One can sometimes turn an autonomous system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y) \tag{9.5}$$

into a first order equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x, y)}{F(x, y)} \tag{9.6}$$

This first order equation can sometimes be solved at least implicitly

$$H(x, y) = c \tag{9.7}$$

This means that the trajectories to (9.5) are the level curves (9.7):

$$0 = \frac{d}{dt} H(x, y) = H_x(x, y) \frac{dx}{dt} + H_y(x, y) \frac{dy}{dt} = H_x(x, y) F(x, y) + H_y(x, y) G(x, y)$$

from which it follows that

$$H_x(x, y) = h(x, y) G(x, y), \qquad H_y(x, y) = -h(x, y) F(x, y)$$

for some function $h(x, y)$. It follows that the critical points for the system (9.5);

$$F(x, y) = G(x, y) = 0 \tag{9.8}$$

are also critical points, or stationary points, for the function $H(x, y)$ in (9.7), i.e.

$$H_x(x, y) = H_y(x, y) = 0. \tag{9.9}$$

Ex 1 Find the trajectories to

$$dx/dt = y, \qquad dy/dt = x$$

Sol We can write

$$\frac{dy}{dx} = \frac{x}{y}$$

Separating variables gives

$$ydy = xdx$$

so

$$\frac{y^2}{2} = \frac{x^2}{2} + C.$$

Hence

$$H(x, y) \equiv y^2 - x^2 = c$$

Alternatively we can solve it with the methods for linear systems which gives

$$x = c_1 e^t + c_2 e^{-t}, \qquad y = c_1 e^t - c_2 e^{-t}$$

This example explains why its called a saddle point, which is because the surface $z = H(x, y)$ close to $x = y$ is shaped like a saddle.

Ex 2 Find the trajectories to

$$dx/dt = y, \qquad dy/dt = -x$$

Sol We can write

$$\frac{dy}{dx} = -\frac{x}{y}$$

Separating variables gives

$$ydy = -xdx$$

so

$$\frac{y^2}{2} = -\frac{x^2}{2} + C.$$

Hence the level curves are circles:

$$H(x,y) \equiv y^2 + x^2 = c$$

Alternatively we can solve it with the methods for linear systems which gives

$$x = A\cos t + B\sin t = R\cos(t-\theta_0), \qquad y = -A\sin t + B\cos t = -R\sin(t-\theta_0).$$

Ex Find the trajectories to the system

$$dx/dt = 4 - 2y, \qquad dy/dt = 12 - 3x^2$$

Sol We have

$$\frac{dy}{dx} = \frac{12-3x^2}{4-2y}$$

so

$$(4-2y)dy = (12-3x^2)dx$$

Hence

$$H(x,y) \equiv 4y - y^2 - 12x + x^3 = c.$$

When we draw these level curves for different values of c it may be helpful to start with the level curves that go through the stationary points of $H(x,y)$ which are the critical points of the system, i.e. $4-2y=0=12-3x^2$, i.e $y=2$ and $x=\pm 2$. At $(-2,2)$ we change variables $x=u-2$, $y=v+2$ then we get the system

$$du/dt = -2v, \qquad dv/dt = 12u - 3u^2$$

The eigenvalues for the linearized system at $(-2,2)$ are hence $\pm i\sqrt{24}$ so the linearized system has a center there. However when the real part of an eigenvalues is zero then the linear system can not be used to determine the stability of the nonlinear system because its to sensitive and we need some room to say that the linear terms dominate the nonlinear terms. Instead we look at the level surfaces close to the critical point

$$H(u-2, v+2) = 32 - v^2 - 6u^2 + u^3$$

For small u and v the level curves

$$v^2 + 6u^2(1-u/6) = c$$

are approximately ellipses $u^2 + 6v^2 = c$, which means that the critical point is a center also for the nonlinear problem.
At $(2,2)$ we change variables $x=u+2$ and $y=v+2$ to obtain

$$H(u+2, v+2) = -10 + 6u^2 - v^2 + u^3$$

This is hence a saddle point.

The pendulum. A nonlinear example that well illustrates these concepts is that of the pendulum. The pendulum consists of a mass m attached to one end of a rigid weightless rod of length L. The other end of the rod is attached to a fixed origin O, around which the rod is free to rotate in a vertical plane. The position of the pendulum is described by an angle θ between the rod and the downward vertical direction, in which direction the gravitational force mg acts. Moreover we assume that we have a damping force or friction $-CL\theta'$ which is proportional to but in opposite direction to the velocity $L\theta'$. Newton's equation $ma = F$ gives

$$mL\frac{d^2\theta}{dt^2} = -CL\frac{d\theta}{dt} - mg\sin\theta$$

With the constants $\omega^2 = \frac{g}{L}$, $\gamma = \frac{C}{m}$ we write it as a system for $x_1 = \theta$ and $x_2 = \theta'$:

$$x_1' = x_2, \qquad x_2' = -\omega^2 \sin x_1 - \gamma x_2$$

The critical points are given by

$$x_2 = 0, \qquad -\omega^2 \sin x_1 - \gamma x_2 = 0,$$

i.e. $x_2 = 0$ and $x_1 = 0$ or $x_1 = \pi$. $x_1 = 0$ corresponds to the downward position. The downward position is stable. However its asymptotically stable only if the damping constant $C > 0$. $x_1 = \pi$ corresponds to the position of the rod straight up, which is an equilibrium if there is no initial velocity, since the forces in this case only acts straight down. However, this equilibrium is highly unstable and the smallest movement from it will lead to large movements of the rod.