Lecture 32: 9.6 Energy and Liapunov functions.

Energy conservation for the Pendulum. Consider the pendulum again:

$$\theta'' + \gamma\theta' + \omega^2 \sin\theta = 0, \qquad \omega^2 = g/L$$

With $x = \theta$ and $y = \theta'$ we get the system

$$x' = y, \qquad y' = -\gamma\, y - \omega^2 \sin x, \qquad \omega^2 = g/L$$

The critical points are $y = 0$ and $\sin x = 0$, i.e. $x = 0, \pi$.

Let us study the critical point at the bottom when $x = 0$. Since for small x, $\sin x \sim x$ the linearized system has matrix

$$\begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix},$$

The characteristic polynomial is

$$\lambda(\lambda+\gamma) + \omega^2 = (\lambda+\gamma/2)^2 + \omega^2 - \gamma^2/4 = 0$$

If the damping is not too large then $\gamma < 2\omega$ and the roots are complex

$$\lambda = -\gamma/2 \pm i\sqrt{\omega^2 - \gamma^2/4}$$

It follows from this that if $\gamma > 0$ the roots have negative real part and therefore the linear system is both stable and asymptotically stable and one can draw the same conclusion for the nonlinear system. However, if $\gamma = 0$ the linear system is only stable and one can not say anything about the stability of the nonlinear system.

Yet, we know from physics that it has to be stable even without damping, i.e. if we start with small a displacement from equilibrium and with small velocity we know that it can't move far away from equilibrium. This follows from energy conservation. The kinetic energy respectively the potential energy are given by

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mL^2(\theta')^2, \qquad U = mgh = mg(1-\cos\theta)$$

The total energy

$$V = T + U = mgL(1-\cos x) + \tfrac{1}{2}mL^2y^2,$$

is conserved in the case of no damping and decreasing if there is damping. In fact, if $(x(t), y(t))$ is a solution to the system above then

$$\frac{d}{dt}V(x,y) = V_x\frac{dx}{dt} + V_y\frac{dy}{dt} = mgL\sin x\; y + mL^2y\big(-\gamma y - \frac{g}{L}\sin x\big) = -\gamma mL^2y^2 \le 0$$

It therefore follows that along trajectory starting at (x_0, y_0) we have for $t \ge 0$

$$V(x,y) \le V(x_0,y_0)$$

Recall that $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + ...$, and because its an alternating series we have $1 - \frac{1}{2}x^2 \le \cos x \le 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$. Hence for $|x| \le 1$

$$\tfrac{1}{4}x^2 \le 1 - \cos x \le \tfrac{1}{2}x^2$$

Hence

$$mgL\tfrac{1}{4}x^2 + \tfrac{1}{2}mL^2y^2 \le mgL\tfrac{1}{2}x_0^2 + \tfrac{1}{2}mL^2y_0^2$$

Since both terms in the energy are positive it follows that both x and y are small if x_0 and y_0 are small. This proves the stability. (Above we assumed that $|\,x| < 1$, which is true initially, and by the existence theorem the trajectories are continuous so this is true for some time and the above proof holds which gives a better bound.)

Liapunov functions. In physical problems often there is an energy that is conserved. However, we didn't need an exact conservation for the above argument to work. For some problems one can construct a function that is decreasing or at least non-increasing along the trajectories and its positive definite, meaning that bounds for it control all components of the system. Let us consider a general system

$$dx/dt = F(x,y), \qquad dy/dt = G(x,y),$$

and suppose that the origin $x = y = 0$ is a critical point.
Suppose that V is a function defined in a set D containing the origin and $V(0,0)=0$. Then V is called **positive definite** if $V(x,y) > 0$ for $(x,y) \neq (0,0)$ (and **positive semidefinite** if only $V(x,y) \geq 0$). Similarly, V is called **negative definite** if $V(x,y) > 0$ for $(x,y) \neq (0,0)$ (and **negative semidefinite** if only $V(x,y) \geq 0$).

Associated with V and the system above we define the function

$$\dot{V}(x,y) = V_x(x,y)F(x,y) + V_y(x,y)G(x,y)$$

Note that for a trajectory of the differential equation above we have

$$\frac{d}{dt}V(x,y) = \dot{V}(x,y)$$

Theorem 1 Suppose that (0.0) is a critical point and suppose that there is a continuous function V with continuous first order derivatives such that V is positive definite. Then if the associated function $\dot{V}$ is negative definite the critical point is asymptotically stable. If $\dot{V}$ is only negative semidefinite the critical point is stable.
Proof of stability As for the pendulum we conclude that along any trajectory of the system that starts at (x_0, y_0) and ends at (x,y), we have $V(x,y) \leq V(x_0,y_0)$. To show that $(0,0)$ is stable we must show that if $\varepsilon > 0$ is sufficiently small there is a $\delta > 0$ such that $|(x_0,y_0)| < \delta$ implies that $|(x,y)| < \varepsilon$. This statement follows from the fact that $V(x,y) \leq V(x_0,y_0)$ and the following two statements:
(i) Given any $\varepsilon > 0$, there is a $c > 0$ such that $V(x,y) < c$ implies that $|(x,y)| < \varepsilon$.
(ii) Given any $c > 0$ there is a $\delta > 0$ such that $V(x_0,y_0) < c$, if $|(x_0,y_0)| < \delta$.
The second statement is just the continuity of $V(x,y)$ at $(0,0)$ where $V(0,0) = 0$. To prove (i) we argue by contradiction. If (i) is not true then there would exist an $\varepsilon > 0$ and a sequence $(x_n,y_n) \in D$ such that $|(x_n,y_n)| \geq \varepsilon$ but $V(x_n,y_n) \to 0$. We could then choose a subsequence converging to a point $(\overline{x},\overline{y}) \in D$, satisfying $|(\overline{x},\overline{y})| \geq \varepsilon$, and since V is continuous we would also have $V(\overline{x},\overline{y}) = 0$. Since V is positive definite it would follow that $(\overline{x},\overline{y}) = (0,0)$, contradicting that $|(\overline{x},\overline{y})| \geq \varepsilon$.
Proof of asymptotic stability Since $V(x(t),y(t))$ is a decreasing function of t it has lower limit V_0. We claim that $V_0=0$. In fact if not, by the continuity of V there would be a $\delta>0$ such that $|(x,y)| < \delta$ implies that $V(x) \leq V_0$, so we conclude that $|(x(t),y(t))| \geq \delta$, for all t. By negating (i) above applied to $-\dot{V}$ we get that $\dot{V}(x(t),y(t)) \leq -k < 0$, which would imply that $V_0 = -\infty$, which contradicts $V_0 \geq 0$. If $(x(t),y(t))$ did not converge to $(0,0)$ we could choose a sequence of times $t_n \to \infty$ such that $|(x(t_n),y(t_n)| \geq \delta$, for some $\delta > 0$, and so $(x(t_n),y(t_n))$ has a limit $(\overline{x},\overline{y})$. Since V is continuous $V(x(t_n),y(t_n)) \to V(\overline{x},\overline{y})$ so $V(\overline{x},\overline{y}) = V_0 = 0$ so $(\overline{x},\overline{y}) = (0,0)$.
Ex. Show that the following system is asymptotically stable:

$$dx/dt = -\tfrac{1}{2}x^3 + 2xy^2, \qquad dy/dt = -y^3$$

Sol The system is asymptotically stable since, if $V(x,y) = x^2 + cy^2$ with $c > 2$

$$\dot{V} = -x^2 + 4x^2y^2 + c2y^4 = -(x^2 - 2y^2)^2 - (2c-4)y^4 < 0, \quad \text{for} \quad (x,y) \neq (0,0),$$

The strict inequality follows since if its 0 then $y = 0$ and $x^2 - 2y^2 = 0$ so also $x = 0$.

Appendix: Proof of the Linearization theorem. The Linearization theorem says that as long as no eigenvalue has zero real part the linearization close to a critical point determines the stability of the nonlinear system (i.e. if it is stable and/or asymptotically stable or unstable). The proof in the two dimensional case reduced to by a change of coordinates to either of the following

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \begin{bmatrix} a & \epsilon \\ 0 & a \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

In either case it reduces to that

$$V(\mathbf{x}) = |\mathbf{x}|^2$$

is a Liapunov function.