Lecture 6: 2.4 Difference between linear and nonlinear differential equations. For linear equations we have the following existence theorem:

Th 1 Suppose that p and g are continuous functions on an open interval $I : \alpha < t < \beta$ containing t_0. Then there is a unique solution to the differential equation

$$\frac{dy}{dt} + p\,(t)\,y = g(t), \qquad t \in I, \qquad y(t_0) = y_0 \tag{2.4.1}$$

In fact, we have already derived a formula for the general solution by multiplying by the integrating factor μ:

$$y = \frac{1}{\mu(t)}\Big(y_0 + \int_{t_0}^{t} \mu(s)g(s)\,ds\Big), \qquad \mu(t) = \exp\Big(\int_{t_0}^{t} p(s)\,ds\Big)$$

However, the derivation assumed that the functions p and g were continuous or else the function above might not be differentiable. In fact we have the following counterexample.

Ex Find the solution to

$$t\,y' + y = 0, \qquad y(1) = 2$$

Sol First we write it in the form (2.4.1):

$$y' + \frac{1}{t}y = 0$$

Multiplying by the integrating factor

$$\mu e^{\int t^{-1}\,dt} = e^{\ln|t|} = |\,t|$$

gives

$$t\,y' + y = 0.$$

Okey, so we got back the equation we started so it seems like we haven't achieved anything. However, the method of multiplying by the integrating factor always makes it so that the left hand side is the derivative of a product:

$$\frac{d}{dt}\big(t\,y\big) = 0$$

This has the general solution

$$t\,y = C$$

or

$$y = \frac{C}{t}$$

Putting in the initial condition $y(1) = C/1 = 2$, gives $C = 2$ so

$$y = \frac{2}{t}$$

This is indeed a solution but note that it tends to infinity when $t \to 0$. What went wrong? The assumptions of the theorem are not satisfied when $t = 0$ since $p(t) = 1/t$ is not continuous when $t = 0$.

The existence theorem for the general nonlinear equation is slightly different

Th 2 Suppose that f and $\partial f/\partial y$ are continuous functions on an open rectangle $R : \alpha < t < \beta, \gamma < y < \delta$ containing (t_0, y_0). Then in some interval $I : t_0 - h < t < t_0 + h$ there is a unique solution to the differential equation

$$\frac{dy}{dt} = f(t,y), \qquad t \in I, \qquad y(t_0) = y_0 \tag{2.4.2}$$

Note the difference with the linear case, that the solution might only exist in some smaller interval I around t_0: $\gamma < t_0 - h < t < t_0 + h < \delta$, for some $h > 0$, in fact:

Ex Find the solution to the differential equation

$$y' = y^2, \qquad y(0) = 1$$

Sol If we separate the variables

$$y^{-2} dy = dt$$

and integrate we get

$$y^{-1} = -t + C$$

or

$$y = \frac{1}{C - t}$$

Since $y(0) = 1/C = 1$ we get

$$y = \frac{1}{1 - t} \tag{2.4.5}$$

This is a solution when $t < 1$, but it goes to infinity when $t \to 1$ even though the right hand side of (2.4.4) is a smooth function for any t.

The phenomena in the previous example that the solution goes to infinity is called **blow-up** and it is typical for nonlinear differential equations. Another thing that can go wrong for a nonlinear differential equation with a right hand side that is not differentiable is uniqueness:

Ex Find the solution(s) to

$$y' = y^{1/3}, \qquad y(0) = 0$$

Sol If we separate the variables

$$y^{-1/3}\, dy = dt$$

and integrate

$$\frac{3}{2} y^{2/3} = t + c$$

or

$$y = \Big(\frac{2}{3}(t + c)\Big)^{3/2}$$

and if we substitute the initial condition $y(0) = (2c/3)^{3/2} = 0$ we get $c = 0$ so

$$y = \Big(\frac{2}{3}t\Big)^{3/2}$$

However, another solution is

$$y = 0$$

Since there are two different solutions we don't have uniqueness. This is due to that the derivative of the right hand side $y^{1/3}$ is not continuous when $y = 0$.

2.8 The proof of the existence theorem. The proof of the existence theorem is outlined in the problems to section 2.8. I advice anyone interested in math to go through it. The first idea is to rewrite the equation (2.4.1) as

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

As a first approximation to the solution we put

$$y_0(t) = y_0$$

and then successively

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds, \qquad n \geq 0$$

It turns that this $y_n(t) \to y(t)$, converges as $n \to \infty$, if t is sufficiently small.