Math 631 Partial Differential Equations I: Linear Equations

Lecture 1: Introduction.

1.1 Definition. A Partial Differential Equation (PDE) of *order* k for a function $u(x)$ of $x \in \mathbf{R}^n$ is an equation involving u and its derivatives up to order k

$$F(x, u(x), \partial u(x), \ldots, \partial^k u(x)) = 0 \tag{1.1}$$

Here $\partial^k u$ stands for the jet of all partial derivatives $\partial^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u$, of order $k = |\,\alpha| = \alpha_1 + \cdots + \alpha_n$. The functions u and F may also be vector valued in which case its called a *system* of partial differential equations. A PDE is called *linear* if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha u(x) = f(x) \tag{1.2}$$

1.2 Examples. Partial Differential equations arise in e.g. physics and geometry:

Linear. Laplace equation:

$$\triangle u = \sum_{i=1}^n \partial_{x_i}^2 u = 0$$

Heat equations

$$\partial_t u - \triangle u = 0$$

Wave equation

$$\Box u = \partial_t^2 u - \triangle u = 0$$

Schroedinger equation

$$i \partial_t u + \triangle u = 0$$

Transport equation

$$\partial_t u + b^i \partial_{x_i} u = 0$$

Ordinary differential equation

$$\partial_t u + Au = 0$$

Nonlinear equations Burgers' equation

$$\partial_t u + u \partial x u = 0$$

Minimal surface equation

$$\sum_{i=1}^n \partial_{x_i} \Big(\frac{\partial_{x_i} u}{(1 + |\partial u|^2)^{1/2}} \Big) = 0$$

Linear Systems Maxwell's equations

$$\begin{cases} E_t = \mathbf{curl}\, B \\ B_t = -\mathbf{curl}\, E \\ \operatorname{div} B = \operatorname{div} E = 0 \end{cases}$$

Nonlinear systems Euler's equations of an incompressible fluid

$$\begin{cases} \partial_t u_i + \sum_{k=1}^n u_k \partial_{x^k} u_i = -\partial_i p \\ \sum_{i=1}^n \partial_{x^i} u_i = 0 \end{cases}$$

Einstein's vacuum equations of general relativity for the metric tensor $g_{\alpha\beta}$, $\alpha, \beta = 0, 1, 2, 3$, of space time is that the Ricci curvature vanishes:

$$R_{\mu\nu}(g) = 0$$

which in harmonic coordinates becomes a system of nonlinear wave equations

$$\Box_g g_{\mu\nu} = F_{\mu\nu}(g, \partial g), \qquad \Box_g = \sum_{\alpha,\beta=0,1,2,3} g^{\alpha\beta} \partial_{x^\alpha} \partial_{x^\beta}$$

Evolution equations. The wave, heat, Schroedinger, transport equations and the ordinary differential equations are *evolution* equations describing evolving phenomena. For evolution equations we want to find a solution for future times from the knowledge of initial conditions.

Stationary equations. Laplace equation is a *stationary equation.* For stationary equations we want to find a solution in the interior of a domain from boundary conditions.

1.3 Strategies for Solving PDE's.

Linear PDEs can be solved more or less explicitly, in particular if the coefficients a_α are constants.

For nonlinear equations we can in general not find an explicit solution but instead we just ask if the problem is *well posed*, i.e. if:
(a) the problem has a solution,
(b) the solution is unique,
(c) the solution depends continuously on data in a certain class.

For nonlinear equations one can usually prove local existence of a solution but the solution might but the solution might blow up after some time.

The ordinary differential equation.

$$\frac{d}{dt} u(t) = 0 \tag{1.1}$$

has the general solution

$$u(t) = c$$

where c is a constant determined from the initial conditions

$$u(0) = c \tag{1.2}$$

The initial value problem (1.1)-(1.2) has a unique solution. Similarly, the initial value problem

$$\frac{d^2}{dt^2} u(t) = 0$$

where

$$u(0) = a, \quad \dot{u}(0) = b$$

has the unique solution

$$u(t) = at + b$$

This last equation describes a moving particle. You need to know both the initial position and the velocity to determine its path. This is Newton's law; mass times acceleration is equal to force.

The initial value problem (IVP) for the simplest linear equations of one space variable.

2.1 The transport equation.

$$u_t(t,x) + cu_x(t,x) = 0 \tag{1.3}$$

This equation just says that u is constant in the direction $(1, c)$, i.e. u is constant along the *characteristic lines* $x - ct = \xi$. In fact

$$\frac{d}{dt}u(t, ct + \xi) = \big(u_t + cu_x\big)(t, ct + \xi) = 0$$

It follows that

$$u(t,x) = f(\xi) = f(x - ct)$$

for some function f. This formula represents the general solution. Note that the solution at time t is the data at time 0 translated the distance ct along the x-axis. The solution is determined uniquely by posing the initial condition

$$u(0,x) = f(x) \tag{1.4}$$

Conversely the initial value problem (1.3)-(1.4) has a unique solution give above.

The solution is a wave being transported at a speed c.

Problem 1.1 Problem 2.5.1 in Evans.

On the other if we in general try to solve

$$au_t + bu_x = 0, \qquad u(0,x) = f(x)$$

we see that it only works if $a \neq 0$, i.e. if the problem is *non-characteristic*. If $a = 0$ and $b \neq 0$ then the first equation says that $u_x = 0$ which contradicts the second equation unless $f'(x) = 0$.

2.4.1a The wave equation.

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0 \tag{1.5}$$

has the general solution

$$u(t,x) = v(x + ct) + w(x - ct) \tag{1.6}$$

for some functions v and w since $(\partial_t \pm \partial_x)h(x \mp ct) = 0$. Note that the solution at time t consist of two waves one traveling to the right and one traveling to the left, both with speed c.

The initial value problem for (1.5) with initial data

$$u(0,x) = f(x), \quad u_t(0,x) = g(x) \tag{1.7}$$

has the solution

$$u(t,x) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \tag{1.8}$$

Problem 1.2 Prove that (1.8) gives the solution to the initial value problem (1.5),(1.7). (There are really two parts to this. First proving that (1.8) is a solution to the initial value problem and second that this is the only solution. If you use (1.6) for the second part prove it.)

The evolution equations and Fourier series. Let us consider the simplest case of solving the linear wave equations on a circle:

$$\partial_t^2 u - \partial_x^2 u = 0, \qquad u(0,x) = f(x), \quad u_t(0,x) = g(x), \tag{2.1}$$

were data are assumed to be periodic $f(x+2\pi) = f(x)$ and $g(x+2\pi) = g(x)$. We we are looking for solution $u(t,x)$ that is periodic in space $u(t,x+2\pi) = u(t,x)$. (This is a simplified version of looking for solutions to the boundary problem with boundary conditions $u(t,0) = u(t,2\pi) = 0$, which is the equation of a string.) Periodic functions can be expanded in a Fourier series for each fixed time t

$$u(t,x) = \sum_{k=-\infty}^{\infty} c_k(t)\, e^{ikx}, \tag{2.2}$$

with coefficients c_k depending on the time. This approach would require that we can expand initial data f and g in a Fourier series:

$$f(x) = \sum_{k=-\infty}^{\infty} a_k\, e^{ikx}, \tag{2.3}$$

There are two different ways in which (2.2) hold and the sum converges. We can talk about that the sum converge point wise for all x to $f(x)$ or that it converges in L^2. It is a theorem that the sum converges point wise to $f(x)$ if $f(x)$ and $f'(x)$ are continuous (in fact, they only need to be piece wise continuous). This theorem can be found in undergraduate differential equation books with boundary value problems. On the other hand equality also hold in L^2 if $f(x)$ is only in L^2 and then the convergence of the sum is in L^2 sense. This statement can be found in the undergraduate Linear Algebra books. The proof of that the Fourier series converges to the function can be found e.g. on Wikipedia or in the undergraduate PDE book by Strauss.