Lecture 14: Analytic Solutions. For the first part below we are following Section 4.6 in Evans but for the proof of convergence we are following Taylor.

The simplest pde

$$\partial_t u(t,x) = -c\partial_x u(t,x), \quad u(0,x) = g(x) \tag{1}$$

can be solved in the class of real analytic solutions. $g(x)$ is called real analytic if g is infinitely differentiable and for each $x_0 \in \mathbf{R}$ there is a $\delta > 0$ such that the power series

$$\sum_{k=0}^{\infty} \frac{g^{(k)}(x_0)}{k!}(x-x_0)^k$$

converges for $|x - x_0| < \delta$. This is then equivalent to that $g(x)$ can be extended to a holomorphic function $g(z)$ for $z \in \mathbf{C}$ satisfying $|z - x_0| < \delta$. This is also equivalent to that

$$|g^{(k)}(x_0)| \le C\lambda^k k!$$

for some constants C and λ. In order to find analytic solutions of (1) we expand in a power series

$$u(t,x) = \sum_{k=0}^{\infty} \frac{\partial_t^k u(0,x)}{k!} t^k$$

Since by (1) $\partial_t u = -c\partial_x u$ and $\partial_t^2 u = \partial_t(-c\partial_x u) = -c\partial_x \partial_t u = (-c\partial_x)^2 u$ and so on $\partial_t^k u = (-c\partial_x)^k u$ we get that

$$u(t,x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(x)}{k!}(-ct)^k$$

converges for $|ct| < \delta$ is g is real analytic. Moreover the sum is equal to $g(x - ct)$.

Now, this simple procedure might not always work since in general. Consider the heat equation.

$$\partial_t u(t,x) = -c\partial_x^2 u(t,x), \quad u(0,x) = g(x)$$

The same procedure would give

$$u(t,x) = \sum_{k=0}^{\infty} \frac{g^{(2k)}(x)}{k!}(-ct)^k$$

which doesn't converge if we take say $g(x) = e^{x^2}$. Also other, more serious problems arises if try to solve the pde

$$(a\partial_t + b\partial_x)u(t,x) = 0, \quad u(0,x) = g(x)$$

when $a = 0$. Then the PDE gives a compatibility condition on initial data; it must satisfy $g'(x) = 0$. Furthermore, we can not calculate $\partial_t u(0,x)$ from initial data so there are no conditions that the time derivatives have to satisfy and hence we do not have uniqueness even if we have existence. The problem arises from that the vector field $a\partial_t + b\partial_x$ and hence its integral curves are tangential to the surface $\{(t,x);\, t = 0\}$ where Cauchy data are posed. if $a = 0$ we say that the surface is characteristic.

Having this simple type of problems in mind let us now proceed to find out what the picture is in general.

Noncharacteristic surfaces

Consider a general quasilinear PDE in $\mathbf{R}^N$:

(1)
$$\sum_{|\alpha|=m} a_\alpha(D^{m-1}u,...,u,x)\, D^\alpha u + a_0(D^{m-1}u,...,u,x) = 0, \quad \text{where} \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial_{x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial_{x_n}^{\alpha_n}}$$

Let us assume that Γ is a smooth, $(n-1)$ dimensional hypersurface. Let $\nu(x^0) = \nu = (\nu_1,...,\nu_n)$ be the unit normal to Γ at a point $x^0 \in \Gamma$. Define the j:th normal derivative to be

$$\frac{\partial^j u}{\partial \nu^j} = \left(\sum_{k=1}^n \nu_k \frac{\partial}{\partial x_k} \right)^j u = \sum_{|\alpha|=j} (D^\alpha u)\, \nu^\alpha, \quad \text{where} \quad \nu^\alpha = \nu_1^{\alpha_1} ... \nu_n^{\alpha_n}$$

The Cauchy problem is then to find a function u solving the (1), subject to the boundary conditions

$$u = g_0, \ \frac{\partial u}{\partial \nu} = g_1, ..., \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = g_{k-1}, \quad \text{on} \quad \Gamma \tag{2}$$

We now pose the basic question: Assuming that u is a smooth solution to (1) do the conditions (2) allow us to compute all partial derivatives of u along Γ? This must certainly be so, if we are ever going to be able to calculate the terms of the power series for u.

Let us first examine the case when Γ is the plane $\{x_n = 0\}$. Then $\nu = (0,...,0,1)$ and hence the Cauchy data (2) becomes

$$u = g_0, \ \frac{\partial u}{\partial x_n} = g_1, ..., \frac{\partial^{m-1} u}{\partial_n^{m-1}} = g_{m-1} \quad \text{on} \quad \{x_n = 0\} \tag{3}$$

Which further partial derivatives of u can we compute along Γ? First, note that since $u = g_0$ on all of Γ we can differentiate tangentially, that is, with respect to x_i, $i = 1,...,n-1$, to obtain

$$D^\alpha u = D^\alpha g_0, \quad \text{on} \quad \Gamma, \quad \text{if} \quad \alpha_n = 0$$

Similarly

$$D^\alpha u = D^{\alpha'} g_k, \quad \text{on} \quad \Gamma, \quad \text{if} \quad \alpha_n = k \le m-1, \quad \alpha' = (\alpha_1,...,\alpha_{n-1},0)$$

The difficulty arise, when we try to calculate

$$\frac{\partial^m u}{\partial x_n^m}$$

Here, we try to use the PDE (1). If $a_{(0,...,0,m)} \neq 0$ the we can solve for

$$\frac{\partial^m u}{\partial x_n^m} = -\frac{1}{a_{(0,...,0,m)}} \left(\sum_{|\alpha|=m,\, \alpha \neq (0,...,0,m)} a_\alpha D^\alpha u + a_0 \right) \tag{4}$$

Everything in the right hand side contains at most $m-1$ derivatives with respect to x_n so it can be calculated in terms of the Cauchy data. Consequently we can now compute also $D^\alpha u$ for $\alpha_n = m$ on Γ, provided that $a_{(0,\dots,0,m)} \neq 0$. We say that Γ is *noncharacteristic* for the PDE (1), if $a_{(0,\dots,0,m)} \neq 0$ on Γ. On the other hand if $a_{(0,\dots,0,m)} = 0$ the in general the PDE and Cauchy data can not be simultaneously satisfied unless Cauchy data satisfies a compatibility condition $\sum_{|\alpha|=m,\, \alpha\neq(0,\dots,0,m)} a_\alpha D^\alpha u + a_0 = 0$ Now given that $a_{(0,\dots,0,m)} \neq 0$ can we now calculate higher order partial derivatives? The answer is yes, since we can obtain all higher order derivatives $D^\alpha u$, with $\alpha_n \geq m$ by differentiating the PDE (4).

Definition 1. We say that the hypersurface Γ is *noncharacteristic* for the PDE (1) if

$$\sum_{|\alpha|=m} a_\alpha \nu^\alpha \neq 0, \quad \text{on} \quad \Gamma \tag{5}$$

If Γ is *noncharacteristic* let us now see that we can compute all partial derivatives of u. This is proven by transforming to the previous case of Cauchy data on $\{x_n = 0\}$. First let us choose any point $x^0 \in \Gamma$. Then we have diffeomorphisms $\Phi, \Psi : \mathbf{R}^n \to \mathbf{R}^n$ so that $\Psi = \Phi^{-1}$ and in a neighborhood of x^0 Γ is given by $x = \Psi(y_1, \dots, y_{n-1}, 0)$. Let $v(y) = u(\Psi(y))$ and hence $u(x) = v(\Phi(x))$. Then v satisfies a PDE

$$\sum_{|\beta|=m} b_\alpha D^\beta v + b_0 = 0 \tag{6}$$

We are going to prove that $b_{(0,\dots,0,m)} \neq 0$ if (5) is satisfied. Since $u(x) = v(\Phi(x))$ we obtain

$$D^\alpha u = \frac{\partial^m v}{\partial y_n^m}(D\Phi^n)^\alpha + \left\{\text{terms not involving } \frac{\partial^m v}{\partial y_n^m}\right\}$$

if $|\alpha| = m$. Thus it follows from (1) that

$$0 = \sum_{|\alpha|=m} a_\alpha D^\alpha u + a_0 = \sum_{|\alpha|=m} a_\alpha (D\Phi^n)^\alpha \frac{\partial^m v}{\partial y_n^m} + \left\{\text{terms not involving } \frac{\partial^m v}{\partial y_n^m}\right\}$$

and so

$$b_{(0,\dots,0,m)} = \sum_{|\alpha|=m} a_\alpha (D\Phi^n)^\alpha$$

Since $D\Phi^n$ is parallel to ν on Γ. Consequently $b_{(0,\dots,0,m)}$ is a nonzero multiple of the term (5).

Cauchy-Kawalevsky theorem

The Cauchy-Kowalewsky theorem, in the linear case, asserts the local existence of a real analytic solution to the "Cauchy problem"

$$
\begin{aligned}
\frac{\partial^m u}{\partial t^m} &= \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} A_{j\alpha}(t,x) \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^j u}{\partial t^j} + f(t,x) \\
u(t_0, x) &= g_0(x), \dots, \partial_t^{m-1} u(t_0, x) = g_{m-1}(x)
\end{aligned} \tag{1.1}
$$

in a neighborhood of (t_0, x_0) given that $A_{j\alpha}(t,x)$ and $f(t,x)$ are real analytic in a neighborhood of (t_0, x_0) and $g_j(x)$ are analytic in a neighborhood of x_0. Without loss of generality we may assume that $(t_0, x_0) = (0, 0)$.

Any system of the form (1.1) can be converted into a first order system:

$$
\partial_t u = L(t,x)\partial_x u + L_0(t,x)u + f, \quad u(0,x) = g(x) \tag{1.2}
$$

where $u = (u_1, \dots, u_N)$ and $L(t,x)\partial_x = \sum_{k=1}^n L_k \partial/\partial_{x^k}$. Here $L_j(t,x)$ are $N \times N$ matrices with analytic elements and f and g are vectors with real analytic components.

Problem 1: Show that one can convert (1.1) into a system of the form (1.2).

If we differentiate (1.2) we obtain

$$
\partial_t^{j+1} u = \sum_{\ell=0}^{j} \binom{j}{\ell} \Big((\partial_t^{j-\ell} L)\partial_x \partial_t^\ell u + (\partial_t^{j-\ell} L_0)\partial_t^\ell u \Big) + \partial_t^j f \tag{1.3}
$$

In particular, this inductively gives $\partial_t^{j+1} u(0,x)$ uniquely so we have at most one analytic solution. On the other hand if we can use (1.3) to get sufficiently good estimates for $u_{j+1}(x) = \partial_t^{j+1} u(0,x)$ so that the power series

$$
u(t,x) = \sum_{j=0}^{\infty} \frac{u_j(x)}{j!} t^j \tag{1.4}
$$

converges for (t,x) close to $(0,0)$ then (1.4) gives a solution to (1.2). To be more precise, set $u_0(x) = g(x)$ and define $u_{j+1}(x)$ inductively by

$$
u_{j+1} = \sum_{\ell=0}^{j} \binom{j}{\ell} \Big((\partial_t^{j-\ell} L)\partial_x u_\ell + (\partial_t^{j-\ell} L_0) u_\ell \Big) + \partial_t^j f \tag{1.5}
$$

Since $g_j(x)$, and $L_j(t,x)$ $f(t,x)$ are real analytic we can extend them to holomorphic functions for x in a neighborhood of 0 in $\mathbf{C}^n$. We keep t real for now. Without loss of generality we may assume that $g_j(z)$, and $L_j(t,z)$ $f(t,z)$ are holomorphic in a neighborhood of the closed unit ball $\overline{B} \in \mathbf{C}^n$, with real analytic dependence on t for $|t| \le 1$: More specifically, we will assume that

$$
\|g\|_{L^\infty(B)} \le C_2, \quad \sum_{k=0}^{n} \|\partial_t^m L_k(0)\|_{L^\infty(B)} \le C_1 \lambda^m m!, \quad \|\partial_t^m f(0)\|_{L^\infty(B)} \le C_2 \mu^m m! \tag{1.6}
$$

for some constants C_1, C_2 and λ, μ.

Problem 2: Why may we assume that these functions are holomorphic in a ball of radius 1?

Let $\mathcal{H}_j$ be the Banach space of holomorphic functions on the (open) unit ball B having the property that

$$N_j(u) = \sup_{z\in B} \delta(z)^j |u(z)| < \infty, \tag{1.7}$$

where $\delta(z) = 1 - |z|$ is the distance from z to ∂B. We have the following properties for these norms

$$N_j(u) \le N_{j-1}(u) \le \ldots \le N_0(u) = \|u\|_{L^\infty(B)} \tag{1.8}$$

$$N_{j+k}(uv) \le N_j(u)\, N_k(v) \tag{1.9}$$

$$N_{j+1}(\partial_x u) \le \gamma(j+1) N_j(u) \tag{1.10}$$

(1.8)-(1.9) are trivial but the proof of (1.10) is longer so we postpone it to later.

We will inductively obtain estimates for $N_j(u_j)$. From (1.5) we obtain

$$\begin{aligned} &N_{j+1}(u_{j+1}) \le \\ &\sum_{\ell=0}^{j} \binom{j}{\ell} \Big(N_{j-\ell}\big(\partial_t^{j-\ell} L(0)\big)\, N_{\ell+1}(\partial_x u_\ell) + N_{j-\ell}\big(\partial_t^{j-\ell} L_0(0)\big)\, N_\ell(u_\ell) \Big) + N_{j+1}\big(\partial_t^j f(0)\big) \\ &\qquad \le \gamma(j+1) \sum_{\ell=0}^{j} \binom{j}{\ell} \Big(\sum_{k=0}^{n} N_{j-\ell}\big(\partial_t^{j-\ell} L_k(0)\big) N_\ell(u_\ell) \Big) + N_{j+1}\big(\partial_t^j f(0)\big) \end{aligned} \tag{1.11}$$

By (1.6)

$$\sum_{k=0}^{n} N_m\big(\partial_t^m L_k(0)\big) \le C_1 \lambda^m m!, \quad N_m\big(\partial_t^m f(0)\big) \le C_2 \mu^m m! \tag{1.12}$$

By, if necessary making λ and μ larger we may assume that

$$\mu = 2\lambda, \quad \mu \ge 2\gamma C_1 + 1 \tag{1.13}$$

Now, our inductive hypothesis on u_ℓ is that there exists constants C_2 and μ such that

$$N_\ell(u_\ell) \le C_2 \mu^\ell \ell!, \quad 0 \le \ell \le j. \tag{1.14}$$

The $\ell = 0$ case follows from our hypothesis on $g(x)$. Substitution of (1.12) and (1.14) into (1.11) gives

$$N_{j+1}(u_{j+1}) \le \gamma C_1 C_2 (j+1)! \sum_{\ell=0}^{j} \lambda^{j-\ell} \mu^\ell + C_2 \mu^j (j+1)! \tag{1.15}$$

Using (1.13) we see that $\sum_{\ell=0}^{j} \lambda^{j-\ell}\mu^\ell \le 2\mu^j$ and

$$N_{j+1}(u_{j+1}) \le C_2 (j+1)!\, (2\gamma C_1) \mu^j + C_2 \mu^j (j+1)! \le C_2 \mu^{j+1} (j+1)! \tag{1.16}$$

This completes the induction.

Let B be the unit ball in $\mathbf{R}^n$ and set

$$N_j(u) = \sup_{x \in B} \delta(x)^j |u(x)|,$$

where $\delta(x) = 1 - |x|$ is the distance from x to ∂B. We need the following lemma:

Lemma 1.1. *Suppose that u is harmonic in B. Then there is constant γ_n depending only on the dimension n such that*

$$N_{j+1}(\partial_k u) \leq \gamma_n (j+1) N_j(u)$$

Lemma 1.2. *Let $B_\rho(x)$ be the ball of radius ρ centered at x in $\mathbf{R}^n$ and suppose that u is harmonic in $B_\rho(x)$. Then there is constant C depending only on the dimension n such that*

$$|\partial_k u(x)| \leq \frac{C}{\rho} \sup_{y \in B_\rho(x)} |u(y)|$$

Proof of Lemma 1.2. Since the inequality is invariant under translation and dilations we may assume that $x = 0$ and $\rho = 1$. The solution of $\triangle u = 0$, with $u = g$ on ∂B is given by Poisson's formula:

$$u(x) = C_n \int_{\partial B} \frac{1 - |x|^2}{|x - y|^n} g(y)\, dS(y)$$

If we differentiate this expression with respect to x_k and put $x = 0$ we get

$$\partial_k u(0) = C_n \int_{\partial B} \frac{-n y_k}{|y|^{n+2}} g(y)\, dS(y)$$

from which the lemma follows.

Proof of Lemma 1.1. Let $x \in B$ and let $B_\rho(x) \subset B$ be a ball of radius $\rho = \beta\delta(x)$, where $\beta < 1$. Since for $y \in B_\rho(x)$ we have $\delta(y) \geq \delta(x) - \rho = (1 - \beta)\delta(x)$ it follows from Lemma 1.2 that

$$\delta(x)^{j+1} |\partial_k u(x)| \leq \frac{C}{(1 - \beta)^j \beta} \sup_{y \in B_\rho(x)} \delta(y)^j |u(y)|$$

If we pick $\beta = 1/(j+1)$ we get the lemma since $\lim_{j \to \infty} (1 - 1/(j+1))^j = e$.