Lecture 2: Convergence of Fourier series. Suppose that $f(x)$ is a period function $f(x+2\pi) = f(x)$. Suppose also that $f(x)$ is either in C^1 or L^2. Then the Fourier series converges uniformly respectively in L^2 to $f(x)$:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

where the Fourier coefficients are given by

$$c_k = \frac{1}{2\pi} \int_{\pi}^{\pi} f(y)\, e^{-iky}\, dy.$$

In order to prove this let us first is to phrase it in a more general abstract setting. Let us introduce the inner product and the norm

$$(f,g) = \frac{1}{2\pi} \int_{\pi}^{\pi} f(x)\overline{g(x)}\, dx, \qquad \|f\| = \sqrt{(f,f)}.$$

Suppose that $\{X_k(x)\}_{k=1}^{\infty}$ is an orthonormal family of functions i.e.

$$(X_m, X_m) = \delta_{mn} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n \end{cases}$$

Let W_N be the subspace spanned by all functions of the forms $\sum_{k=-N}^{N} c_k X_k$. We want to find the function $f_N = \sum_{k=-N}^{N} c_k X_k \in W_N$ that best approximates f, in the sense that it makes the norm

$$\|f - f_N\|$$

as small as possible. We have

$$\|f - f_N\|^2 = (f - f_N, f - f_N) = \|f\|^2 + \|f_N\|^2 - (f, f_N) - (f_N, f)$$

$$= \|f\|^2 + \sum_{k=-N}^{N} |c_k|^2 - c_k\overline{(f, X_k)} - \overline{c_k}(f, X_k) = \|f\|^2 - \sum_{k=-N}^{N} |(f, X_k)|^2 + \sum_{k=-N}^{N} |c_k - (f, X_k)|^2$$

This sum is minimized for

$$c_k = c_k(f) = (f, X_k).$$

for all k. Furthermore, with this choice of c_k we conclude that

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 \leq \|f\|^2.$$

This is called Bessel's inequality. Moreover, there is equality if and only if f_N converges to f in L^2 norm.

We now leave the setting and get into the analytical detailed estimates. We have

$$f_N(x) = \sum_{k=-N}^{N} c_k e^{ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-N}^{N} e^{ik(x-y)} f(y)\, dy$$

Here by the formula for a geometric sum $\sum_{k=0}^{N} a^k = (1-a^{k+1})/(1-a)$;

$$K_N(z) = \sum_{k=-N}^{N} e^{ikz} = \sum_{k=0}^{N} (e^{iz})^k + \sum_{k=0}^{N} (e^{-iz})^k - 1 = \frac{1-e^{i(k+1)z}}{1-e^{iz}} + \frac{1-e^{-i(k+1)z}}{1-e^{-iz}} - 1$$

which can be seen to be equal to

$$K_N(z) = \frac{\sin\left((N+1/2)z\right)}{\sin\left(z/2\right)}$$

On the other hand it also follows from the first expression that since the integral of the exponential vanish we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(z)\, dz = 1$$

Therefore

$$f_N(x)-f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(z)\big(f(z-x)-f(x)\big)\, dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left((N+1/2)z\right) g_x(z)\, dz,$$

where

$$g_x(z) = \frac{f(z-x)-f(x)}{\sin\left(z/2\right)}$$

is continuous if $f \in C^1$. Here $X_N(z) = \sqrt{2}\sin\left((N+1/2)z\right)$, $N = 1, \ldots$ is an orthonormal family of functions $(X_N, X_M) = \delta_{MN}$. Therefore by the previous part

$$\sum_{N=0}^{\infty} |(g_x, X_N)|^2 \le \|g\|^2$$

and if we integrate both sides with respect to x

$$2\sum_{N=0}^{\infty} \|f - f_N\|^2 = \sum_{N=0}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |(g_x, X_N)|^2\, dx \le \|g\|^2$$

Since the sum is bounded it follows that the terms tend to 0 and hence $\|f_N - f\| \to 0$ as $N \to \infty$, i.e. f_N converges to f in L^2.

Summarizing we have proven that if $f \in C^1$ then the Fourier series converges to f in L^2 and moreover

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 = \|f\|_{L^2}^2$$

Moreover by Bessel's inequality the Fourier coefficients for the derivative satisfy

$$\sum_{k=-\infty}^{\infty} |c_k(f')|^2 \leq \|f'\|_{L^2}^2$$

If we integrate by parts we get

$$c_k(f') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-ikx}\,dx = ik\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-ikx}\,dx = ikc_k(f).$$

Therefore

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2(1+k^2) \leq \|f'\|^2 + \|f\|^2.$$

Hence

$$\sum_{k=-\infty}^{\infty} |c_k| \leq \sqrt{\sum \frac{1}{1+k^2}}\sqrt{\sum (1+k^2)|c_k|^2} < \infty$$

It follows that

$$\sup_x |f_N(x) - f(x)| = |\sum_{|k|>N} c_k(f)e^{ikx}| \leq \sum_{|k|>N} |c_k(f)| \to 0, \qquad \text{as } k \to \infty.$$

i.e. f_N converges to f uniformly.

Problem 2.1 Show that there is a constant C such that for all periodic $f \in C^1$ we have

$$\sup_x |f(x)| \leq C(\|f'\| + \|f\|)$$

Problem 2.2 Show that for all periodic $f \in L^2$ we have

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 = \|f\|_{L^2}^2$$

Hint: We have proven that this is true for any such $f \in C^1$. It is possible to approximate a function $f \in L^2$ by a sequence of functions $f_n \in C^1$ tending to f in L^2 norm.