Lecture 3: The evolution equations and Fourier series. Let us consider the simplest case of solving the linear wave equations on a circle:

$$\partial_t^2 u - \partial_x^2 u = 0, \qquad u(0,x) = f(x), \quad u_t(0,x) = g(x), \tag{2.1}$$

were data are assumed to be periodic $f(x+2\pi) = f(x)$ and $g(x+2\pi) = g(x)$. We we are looking for solution $u(t,x)$ that is periodic in space $u(t,x+2\pi) = u(t,x)$. (This is a simplified version of looking for solutions to the boundary problem with boundary conditions $u(t,0) = u(t,2\pi) = 0$, which is the equation of a string.) Periodic functions can be expanded in a Fourier series

$$u(t,x) = \sum_{k=-\infty}^{\infty} c_k(t)\, e^{ikx}. \tag{2.2}$$

If you don't know this fact we can just say that we are looking for solutions of this form. If this is to satisfy the wave equation then we must have

$$\partial_t^2 u - \partial_x^2 u = \sum_{k=-\infty}^{\infty} \big(\ddot{c}_k(t) + k^2 c_k(t)\big) e^{ikx} = 0$$

from which it follows that we must have

$$\ddot{c}_k(t) + k^2 c_k(t) = 0, \tag{2.3}$$

for all k. Solving this linear ordinary differential equation gives

$$c_k(t) = A_k e^{ikt} + B_k e^{-ikt}$$

The constants A_k and B_k are determined by expanding the initial data in Fourier series

$$u(0,x) = f(x) = \sum_{k=-\infty}^{\infty} d_k e^{ikx}, \qquad u_t(0,x) = g(x) = \sum_{k=-\infty}^{\infty} e_k e^{ikx} \tag{2.4}$$

where

$$c_k(0) = A_k + B_k = d_k, \qquad \dot{c}_k(0) = ik(A_k - B_k) = e_k$$

If you don't know the fact that all periodic functions can be expanded in Fourier series we still have proven that the solution to (2.1) is given by (2.2) for all initial data of the form (2.4).

This method works fine also for the other evolution equations:

Problem 2.1: Use Fourier series to find the solution to the initial value problem for the heat equation

$$u_t - u_{xx} = 0, \qquad u(0,x) = f(x) = \sin 3x,$$

Now, it also tells us something about in which class we can expect solutions. By Parsevals formula

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx = \sum_{k=-\infty}^{\infty} |\, c_k|^2, \qquad \text{if} \quad f(x) = \sum c_k e^{ikx}.$$

Formally, this follows from multiplying:

$$\int_0^{2\pi} f(x)\overline{f}(x)\, dx = \sum_{k,\ell=-\infty}^{\infty} \int_0^{2\pi} c_k \overline{c}_\ell e^{i(k-\ell)x}\, dx = \sum_{k,=-\infty}^{\infty} c_k \overline{c}_k 2\pi$$

Since ∂_x corresponds to multiplying the Fourier coefficients by ik we also get

$$\frac{1}{2\pi}\int_0^{2\pi} |\partial_x^j f(x)|^2\, dx = \sum_{k=-\infty}^{\infty} |\, c_k|^2 |k|^{2j}$$

This is called the H^j Sobolev norm of f and denoted by $\|f\|_{H^j}$. As we shall see it is a natural class to seek solutions in. If $u \in H^j([0,2\pi]))$ then $u \in C^{j-1}([0,2\pi])$, the class of $j-1$ times continuously differentiable functions. This follows for $j=1$, since for $0 \leq x \leq y$;

$$u(y) - u(x) = \int_x^y u'(s)\, ds \leq \Big(\int_x^y ds\Big)^{1/2}\Big(\int_0^{2\pi} u'(s)^2\, ds\Big)^{1/2},$$

by Cauchy-Schwarz inequality. Repeating this for derivatives $\partial^{j-1}u$ in place of u gives

$$\|u\|_{C^{j-1}} = \sup_x |\partial^{j-1}u(x)| \leq \Big(\int_0^{2\phi} (\partial^j u(s))^2\, ds\Big)^{1/2} = \|u\|_{H^j}$$

If $u \in H^3$ it follows that $u \in C^2$ so its a classical solution of (2.1), i.e. the derivatives are defined.

For the wave equation the Fourier coefficients are just multiplied by a complex factors which don't change the magnitude. In fact

$$\dot{c}_k(t)^2 + k^2 c_k(t)^2 = \dot{c}(0)^2 + k^2 c_k(0)^2 = e_k^2 + k^2 d_k^2$$

which is seen by taking the derivative of the left hand side and using (2.3):

$$\sum \big(\dot{c}_k(t)^2 + k^2 c_k(t)^2\big) k^{2N} = \big(e_k^2 + k^2 \sum d_k^2\big) k^{2N}$$

or if $\dot{u}$ denotes the time derivative of u:

$$\|\dot{u}(t,\cdot)\|_{H^N}^2 + \|u(t,\cdot)\|_{H^N}^2 = \|g\|_{H^N}^2 + \|f\|_{H^N}^2$$

where $\|u(t,\cdot)\|_{H^k}$ stands for the H^k norm of $u(t,x)$ in the x variables for fixed t. This means that if initial data are in a certain Sobolev space so is the solution be.

For the heat equation its even better and the Fourier coefficients decay more for positive times than initially.

The interesting thing is if we try to solve Laplace equation

$$u_{tt} + u_{xx} = 0$$

with Fourier series, we get

$$c_k(t) = e^{kt} A_k + e^{-kt} B_k$$

so the Fourier modes are exponentially growing if say initially $A_k \sim e^{-\sqrt{|k|}}$, which are the Fourier coefficients of a smooth function. The only way we can get the Fourier coefficients to converge to 0 at positive times is that initial data are real analytic, say $A_k \sim e^{-|k|}$.

Problem 2.2 Show that the backward heat equation

$$u_t = -u_{xx}$$

does not have any solution in Sobolev spaces even if initial data are analytic with Fourier coefficients satisfying

$$c_k \sim e^{-|k|}$$

In general one can think of an evolution equation, e.g. the wave equation, as an infinity dimensional system

$$u_{tt} = Au$$

where A is the operator $A = \triangle$ expressed in the infinite dimensional Fourier bases $\{e^{ikx}\}$ as multiplying the coefficient c_k of e^{ikx} by $-k^2$.

Lecture 3: The Fourier transform. The Fourier transform $\mathcal{F} : f \to \hat{f}$ is defined to be

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx. \tag{3.1}$$

The Fourier transform is invertible, in fact we will prove Fourier's inversion formula:

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, dx \tag{3.2}$$

The Fourier transform makes sense for a very general class of functions and even distributions. However, it is natural to first define it for a more restrictive class and afterwards extend the definition by continuity. This is the Schwartz class $\mathcal{S}$ consisting of all infinitely differentiable functions that are rapidly decreasing:

$$\sup_x |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indices α and β. The importance of this class is that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$. which follows from the following identities for the Fourier transform:

$$\mathcal{F} : \partial_j f(x) \to i\xi_j \hat{f}(\xi), \qquad \mathcal{F} : x_j f(x) \to i\partial_j \hat{f}(\xi) \tag{3.3}$$

(3.3) follows from integrating by parts in (3.1) respectively differentiating below the integral sign. That $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ now follows using (3.1) and integrating by parts

$$\xi^\alpha \partial_\xi^\beta \hat{\phi}(\xi) = \int \xi^\alpha (-i)^{|\beta|} x^\beta e^{-ix\cdot\xi} \phi(x)\, dx = (-1)^{|\alpha|}(-i)^{|\alpha|} \int \partial_x^\alpha \big(x^\beta \phi(x) \big)\, dx$$

which can be bounded by $\sup_x \big|\partial_x^\alpha \big(x^\beta \phi(x)\big)\big| (1+|x|)^{1+n} \int (1+|x|)^{-1-n} dx \leq C$, since $\phi \in \mathcal{S}$.

Note also that by changing variables we get two more simple properties

$$\mathcal{F} : f(ax) \to a^{-n} \hat{f}(\xi/a), \qquad \mathcal{F} : f(x+h) \to \hat{f}(\xi) e^{ih\cdot\xi} \tag{3.4}$$