Lecture 3: The Fourier transform. The Fourier transform $\mathcal{F}: f \to \hat{f}$ is defined to be

$$(3.1) \qquad \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx.$$

The Fourier transform is invertible, in fact we will prove Fourier's inversion formula:

$$(3.2) \qquad f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, dx$$

The Fourier transform makes sense for a very general class of functions and even distributions. However, it is natural to first define it for a more restrictive class and afterwards extend the definition by continuity. This is the Schwartz class $\mathcal{S}$ consisting of all infinitely differentiable functions that are rapidly decreasing:

$$\sup_x |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indices α and β. The importance of this class is that $\mathcal{F}: \mathcal{S} \to \mathcal{S}$. which follows from the following identities for the Fourier transform:

$$(3.3) \qquad \mathcal{F}: \partial_j f(x) \to i\xi_j \hat{f}(\xi), \qquad \mathcal{F}: x_j f(x) \to i\partial_j \hat{f}(\xi)$$

(3.3) follows from integrating by parts in (3.1) respectively differentiating below the integral sign. That $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ now follows using (3.1) and integrating by parts

$$\xi^\alpha \partial_\xi^\beta \hat{\phi}(\xi) = \int \xi^\alpha (-i)^{|\beta|} x^\beta e^{-ix\cdot\xi} \phi(x)\, dx = (-1)^{|\alpha|}(-i)^{|\alpha|} \int \partial_x^\alpha \big(x^\beta \phi(x)\big)\, dx$$

which can be bounded by $\sup_x \big|\partial_x^\alpha \big(x^\beta \phi(x)\big)\big|(1+|x|)^{1+n} \int (1+|x|)^{-1-n} dx \le C$, since $\phi \in \mathcal{S}$.

Note also that by changing variables we get two more simple properties

$$(3.4) \qquad \mathcal{F}: f(ax) \to a^{-n} \hat{f}(\xi/a), \qquad \mathcal{F}: f(x+h) \to \hat{f}(\xi) e^{ih\cdot\xi}$$

The proof of (3.2) uses:

Lemma 3.1.

$$(3.5) \qquad \mathcal{F}: e^{-|ax|^2/2} \to (2\pi)^{n/2} a^{-n} e^{-|\xi/a|^2/2}$$

Proof. Let $\phi(x) = e^{-|x|^2/2}$. Since $(\partial_j + x_j)\phi(x) = 0$ it follows from (4) that $(i\xi_j + i\partial_j)\hat{\phi}(\xi) = 0$, $j = 1, ..., n$. This differential equation has the solution $\hat{\phi}(\xi) = c_n e^{-|\xi|^2/2}$. In fact, if we integrate it for $j = 1$ we get $\hat{\phi}(\xi) = F_2(\xi_2, \cdots, \xi_n) e^{-\xi_1^2/2}$. Plugging this into the same equation gives $(i\xi_k + i\partial_k) F_2(\xi_2, \cdot, \xi_n) = 0$, for $k \ge 2$. Integrating this equation for $k = 2$ gives $F_2(\xi_2, \cdots, \xi_n) = F_3(\xi_3, \cdots, \xi_n) e^{-\xi_2^2/2}$. Repeating this gives that $\hat{\phi}(\xi) = c_n e^{-\xi_1^2/2} \cdots e^{-\xi_n^2/2} = c_n e^{-|\xi|^2/2}$. It therefore only remains to calculate c_n. However, by (3.1) $c_n = \hat{\phi}(0) = \int e^{-|x|^2/2}\, dx$. If $n = 2$ this integral can easily be calculated by introducing polar coordinates $\int_{\mathbf{R}^2} e^{-|x|^2/2}\, dx = 2\pi \int_0^\infty e^{-r^2/2}\, r\, dr = 2\pi$. In general we can write $\int_{\mathbf{R}^n} e^{-|x|^2/2}\, dx = (\int_{\mathbf{R}} e^{-x_1^2/2}\, dx_1)^n$ and $\int_{\mathbf{R}} e^{-x_1^2/2}\, dx_1 = (\int_{\mathbf{R}^2} e^{-|x|^2/2}\, dx)^{1/2}$ so $c_n = (2\pi)^{n/2}$. $\square$

Lemma 3.2. *If $\phi \in \mathcal{S}$ set $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$, then*

$$\int f(x)\,\phi_\varepsilon(x)\,dx = \int f(\varepsilon x)\,\phi(x)\,dx \to f(0)\int \phi(x)\,dx \quad \varepsilon \to 0, \quad \textit{for} \quad f \in \mathcal{S}$$

Proof. Since $|f(\varepsilon x)\,\phi(x)| \le \sup_y |f(y)|\,|\phi(x)|$ the lemma follows from the theorem of Dominated converge. It is also easy to prove directly; since $|f(\varepsilon x)\,\phi(x) - f(0)\phi(x)| \le \varepsilon \sup_y ||y|f'(y)|\,|\phi(x)|$ the difference of the two integrals is bounded by $C\varepsilon$. □

We also have

$$\int \hat{\phi}\psi\,d\xi = \int \phi\hat{\psi}dx, \qquad \phi, \psi \in \mathcal{S} \tag{3.6}$$

In fact, both sides of (6) are equal to the double integral

$$\iint \phi(x)\psi(\xi)\,e^{-ix\cdot\xi}\,dxd\xi$$

It follows from using (3.3) that it suffices to prove (3.2) for $x = 0$ since its translation invariant. Using (3.6) gives

$$\int \hat{\phi}(x)\;\; f(\varepsilon x)\,dx = \int \phi(\varepsilon\xi)\,\hat{f}(\xi)\,d\xi$$

By Lemma 2 we get as $\varepsilon \to 0$

$$\int \hat{\phi}(x)\,dx\,f(0) = \phi(0)\int \hat{f}(\xi)\,d\xi$$

Picking $\phi(x) = e^{-|x|^2/2}$ we get from Lemma 1 and its proof that $\int \hat{\phi}(x)\,dx = (2\pi)^n$ and Fourier's inversion formula (3.2) follows. Using Fourier's inversion formula and (3.6) we get Parseval's formula

$$\int \phi(x)\overline{\psi(x)}\,dx = \frac{1}{(2\pi)^n}\int \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\,d\xi \tag{3.7}$$

In particular;

$$\int |\phi(x)|^2\,dx = \frac{1}{(2\pi)^n}\int |\hat{\phi}(\xi)|^2\,d\xi.$$

We can therefore extend the Fourier transform by continuity to a map $\mathcal{F}:\,L^2 \to L^2$. (This follows since for any $f \in L^2$ one can find a sequence of functions $f_n \in \mathcal{S}$ such that $f_n \to f$ in L^2 and it follows that $\hat{f}_n \in \mathcal{S}$ converges in L^2 to some function $\hat{f}$.) It also follows that

$$\sum_{i=1}^n \int |\partial_i \phi(x)|^2\,dx = \frac{1}{(2\pi)^n}\sum_{i=1}^n \int |\xi_i \hat{\phi}(\xi)|^2\,d\xi = \frac{1}{(2\pi)^n}\int |\xi|^2|\hat{\phi}(\xi)|^2\,d\xi$$

It is now natural to define the Sobolev norms

$$\|\phi\|_{H^s} = \sqrt{\int (1+|\xi|^2)^s|\hat{\phi}(\xi)|^2\,d\xi} \tag{3.8}$$

For integer values of s this corresponds to L^2 norms of derivatives of ϕ, but the norm makes sense and is useful also for real s. This shows that there is a relation between decay of the Fourier transform and regularity of the function.

Problem 3.1 Find the Fourier transform of $e^{-|x|}$, $x \in \mathbf{R}$.

Problem 3.2 Find the inverse Fourier transform of $\sin|\xi|/|\xi|$, $\xi \in \mathbf{R}$.

Lecture 4: Solving initial value problem with the Fourier transform. Recall that the Fourier transform is given by

$$\hat{f}(\xi) = \int f(x) e^{-ix\xi}\, dx \tag{4.1}$$

Let the convolution be defined by

$$K * g(x) = \int K(y) g(x-y)\, dy = \int K(x-y) g(y)\, dy \tag{4.2}$$

It is easy to see that

$$\mathcal{F}:\ f * g \to \hat{f}\hat{g} \tag{4.3}$$

The heat equation. Let us now look on the heat equation

$$\partial_t u(t,x) - \triangle u(t,x) = 0 \tag{4.3}$$

$$u(0,x) = g(x) \tag{4.4}$$

taking the Fourier transform with respect to the space variables only:

$$\hat{u}(t,\xi) = \int u(t,x)\, e^{-ix\cdot\xi}\, dx \tag{4.5}$$

gives

$$\partial_t \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0 \tag{4.6}$$

$$\hat{u}(0,\xi) = \hat{g}(\xi) \tag{4.7}$$

Hence

$$\hat{u}(t,\xi) = e^{-t|\xi|^2} \hat{g}(\xi) \tag{4.8}$$

By Lemma 3.1 with $a = 1/\sqrt{2t}$

$$K_t(x) = \mathcal{F}^{-1}(e^{-t|\xi|^2})(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \qquad t > 0 \tag{4.9}$$

and by (4.3)

$$u(t,x) = K_t * g(x) = \int \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} g(y)\, dy, \qquad t > 0 \tag{4.10}$$

Problem 4.1: Verify directly that $K_t * g(x) \to g(x)$, when $t \to 0$.

We can solve the inhomogeneous problem

$$\partial_t w(t,x) - \triangle w(t,x) = F(t,x)$$

$$w(0,x) = 0$$

using the formula for the homogeneous problem. We claim that

$$w(t,x) = \int_0^t w_s(t,x)\,ds$$

where w_s is the solution of

$$\begin{aligned} \partial_t w_s(t,x) - \triangle w_s(t,x) &= 0 \\ w(s,x) = g_s(x) &= F(s,x) \end{aligned}$$

In fact

$$(\partial_t + \triangle)\int_0^t w_s(t,x)\,ds = w_t(t,x) + \int_0^t (\partial_t + \triangle) w_s(t,x)\,ds = F(t,x)$$

We have

$$w_s(t,x) = K_{t-s} * g_s(x) = \int \frac{1}{(4\pi(t-s))^{n/2}} e^{-|x-y|^2/4(t-s)} F(s,y)\,dy$$

and hence

$$w(t,x) = \int_0^t K_{t-s} * g_s(x)\,ds = \int_0^t \int \frac{1}{(4\pi(t-s))^{n/2}} e^{-|x-y|^2/4(t-s)} F(s,y)\,dy\,ds$$

More generally, the solution to the inhomogeneous problem

$$\begin{aligned} \partial_t v(t,x) - \triangle v(t,x) &= F(t,x) \\ v(0,x) &= g \end{aligned}$$

is linearity given by

$$v(t,x) = u(t,x) + w(t,x) = K_t * g(x) + \int_0^t K_{t-s} * g_s(x)\,ds = \dots.$$

The Schrödinger equation. Let us now consider the Schrödinger equation

$$i\partial_t u(t,x) + \triangle u(t,x) = 0 \tag{4.11}$$
$$u(0,x) = g(x) \tag{4.12}$$

taking the Fourier transform $\hat{u}(t,\xi) = \int u(t,x)\, e^{-ix\cdot\xi}\,dx$ gives

$$i\partial_t \hat{u}(t,\xi) - |\xi|^2 \hat{u}(t,\xi) = 0 \tag{4.13}$$
$$\hat{u}(0,\xi) = \hat{g}(\xi) \tag{4.14}$$

Hence

$$\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{g}(\xi) \tag{4.15}$$

By Formally replacing t by it in (4.9) we get

$$K_t(x) = \mathcal{F}^{-1}(e^{it|\xi|^2})(x) = \frac{1}{(4\pi it)^{n/2}} e^{i|x|^2/4t}, \quad t > 0 \tag{4.16}$$

where we interpret $i^{1/2}$ as $e^{i\pi/4}$, and if we can justify (4.16) we get

$$u(t,x) = K_t * g(x) = \int \frac{1}{(4\pi it)^{n/2}} e^{i|x-y|^2/4t} g(y)\, dy \tag{4.17}$$

Now for $t > 0$ it is easy to see by direct calculation that

$$(i\partial_t + \triangle)K_t = 0 \tag{4.18}$$

But

$$(i\partial_t + \triangle) \int K_t(x-y)\, g(y)\, dy = \int (i\partial_t + \triangle) K_t(x-y)\, g(y)\, dy = 0 \tag{4.19}$$

so (4.17) is a solution of (4.11). However, it still remains to prove that $K_t * g(x) \to g(x)$, when $t \to 0$, which requires stationary phase which we will deal with later on. One can also prove that (4.16) follows from (4.9) by analytic continuation, once we show that that the Fourier transform of K_t is well defined. Since $K_t \notin L^2$ we so far have not defined its Fourier transform.

The wave equation. Let us now look on the wave equation

$$\partial_t^2 u(t,x) - \triangle u(t,x) = 0 \tag{4.20}$$

$$u(0,x) = f(x), \qquad u_t(0,x) = g(x) \tag{4.21}$$

taking the Fourier transform $\hat{u}(t,\xi) = \int u(t,x)\, e^{-ix\cdot\xi}\, dx$ gives

$$\partial_t^2 \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0 \tag{4.22}$$

$$\hat{u}(0,\xi) = \hat{f}(\xi), \qquad \partial_t \hat{u}(0,\xi) = \hat{g}(\xi) \tag{4.23}$$

It is easy to see that this second order ODE has the solution

$$\hat{u}(t,\xi) = \cos(t|\xi|)\, \hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\, \hat{g}(\xi) \tag{4.24}$$

The inverse Fourier transform of $\cos(t|\xi|)$ and $\sin(t|\xi|)/|\xi|$ are so far not defined since these functions are not in L^2, in $\mathbf{R}^n$, if $n \geq 2$. not functions but distributions. In fact the inverse Fourier transform of these functions can not even be defined as a function. Instead we will in the next section define the inverse Fourier transform of these functions as distributions.

Problem 4.2 If $\xi \in \mathbf{R}$ find the inverse Fourier transform of $\cos(t|\xi|) = \cos(t\xi) = \big(e^{it\xi} + e^{-it\xi}\big)/2$ and $\sin(t|\xi|)/|\xi| = \sin(t\xi)/\xi$ and use it to obtain the following integral representation of the solution of (4.20)-(4.21):

$$u(t,x) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy \tag{4.25}$$

Problem 4.3 Show that

$$|\partial_t \hat{u}(t,\xi)|^2 + |\,|\xi|\hat{u}(t,\xi)|^2 = |\hat{g}(\xi)|^2 + |\,|\xi|\hat{f}(\xi)|^2$$

and use it to prove the energy identity

$$\int |\partial_t u(t,x)|^2 + \sum_{i=1}^n |\partial_i u(t,x)|^2\, dx = \int |g|^2 + \sum_{i=1}^n |\partial_i f|^2\, dx$$

Problem 4.4. Show that that the solution to the inhomogeneous wave equation

$$\partial_t^2 w(t,x) - \triangle w(t,x) = F(t,x)$$
$$w(0,x) = 0, \qquad (\partial_t w)(0,x) = 0$$

is given by $w(t,x) = \int_0^t w_s(t,x)\,ds$, where w_s are the solutions to

$$\partial_t^2 w_s(t,x) - \triangle w_s(t,x) = 0$$
$$w_s(s,x) = 0, \qquad (\partial_t w_s)(s,x) = g_s(x) = F(s,x)$$

Some fundamental solutions using the Fourier transform. Let us first consider the transport equation

$$\partial_t u(t,x) + \partial_x u(t,x) = 0, \tag{3.10}$$
$$u(0,x) = g(x) \tag{3.11}$$

Let $\hat{u}(t,\xi) = \int u(t,x) e^{-ix\xi}\,dx$ be the Fourier transform of $u(t,x)$ with respect to x for t fixed. Then

$$\partial_t \hat{u}(t,\xi) + i\xi \hat{u}(t,\xi) = 0, \tag{3.12}$$
$$\hat{u}(0,\xi) = \hat{g}(\xi) \tag{3.13}$$

Solving the PDE (3.10)-(3.11) now reduces to solving the ODE (3.12)-(3.13) for fixed ξ:

$$\hat{u}(t,\xi) = e^{-i\xi t}\hat{g}(\xi) \tag{3.14}$$

Taking the inverse Fourier transform gives

$$u(t,x) = \frac{1}{2\pi}\int e^{ix\xi} e^{-i\xi t}\hat{g}(\xi)d\xi = \frac{1}{2\pi}\int e^{i(x-t)\xi}\hat{g}(\xi)d\xi = g(x-t) \tag{3.15}$$

Note also that by (3.15) $|\hat{u}(t,\xi)| = |\hat{g}(\xi)|$ so by Parseval's formula (3.6) we get the energy identity

$$\int |u(t,x)|^2\,dx = \int |g(x)|^2\,dx \tag{3.16}$$

and more generally $\|u(t,\cdot)\|_{H^s} = \|g\|_{H^s}$. One can now also solve the inhomogeneous problem

$$\partial_t u(t,x) + \partial_x u(t,x) = F(t,x),$$
$$u(0,x) = 0$$

We claim that $u(t,x) = \int_0^t u_s(t,x)\,ds$ where u_s is the solution of

$$\partial_t u_s(t,x) + \partial_x u_s(t,x) = 0,$$
$$u_s(s,x) = g_s(x) = F(s,x)$$

In fact

$$(\partial_t + \partial_x)\int_0^t u_s(t,x)\,ds = u_t(t,x) + \int_0^t (\partial_t + \partial_x) u_s(t,x)\,ds = F(t,x)$$

It follows that is given by a translation of the solution of (3.10)-(3.11) so

$$u_s(t,x) = F\big(s, x-(t-s)\big)$$

so

$$u(t,x) = \int_0^t u_s(t,x)\,ds = \int_0^t F\big(s, x-(t-s)\big)\,ds$$