Lecture 5: Weak Solutions. We have seen that a general solution to the equation

$$u_{tt} - u_{xx} = 0 \tag{5.1}$$

is given by

$$u(t,x) = h(t-x) + k(t+x) \tag{5.2}$$

If $h, k \in C^2$ then (5.2) is C^2 so the equation (5.1) makes sense point wise. This is called a *classical solution* However, the expression (5.2) makes sense if h, k are just integrable functions, and ought to be recognized as a solution of (5.1) in some sense. We will now define a class of *weak solutions* that generalize the concept of classical solutions. Let us therefore start with a classical solution of (5.1) which is in C^2 so the left hand side of (5.1) is continuous. (5.1) is then equivalent to that

$$\iint \big(u_{tt} - u_{xx}\big)\phi \, dx dt = 0, \qquad \text{for all } \phi \in C_0^\infty \tag{5.3}$$

Its clear that (5.1) implies (5.3). Suppose that (5.1) does not hold at some point (t_0, x_0), where the left is say positive. Then since the left of (5.1) is continuous $u_{tt} - u_{xx}$ will be positive in some open set and we can pick a non vanishing test function ϕ with $\operatorname{supp}\phi$ contained in that open set so the left of (5.3) would be strictly positive.

Since ϕ has compact support we can integrate by parts twice without any boundary terms to obtain

$$\iint u\big(\phi_{tt} - \phi_{xx}\big) dx dt = 0, \qquad \text{for all } \phi \in C_0^\infty \tag{5.4}$$

We have shown that any classical solution of (5.1) also satisfies (5.4). We now make the following definition. We say that u is a **weak solution** of (5.1) if u is integrable and (5.4) hold.

Problem 5.1 Show that (5.2) is weak solution of (5.1) if h and k are continuous.

Problem 5.2 Find a formula for the solution of

$$u_{tt} - u_{xx} = F, \quad u(0,x) = f(x), \quad u_t(0,x) = g(x).$$

Distributions. Let $C_0^\infty(\mathbf{R}^n)$ (or $\mathcal{D}$) denote the set of infinitely differentiable functions that have compact support. (The support of a continuous function ϕ is defined by supp $\phi = \{x \in \mathbf{R}^n;\ \phi(x) \neq 0\}$.) The seminorms

$$\rho_{\alpha,K}(\phi) = \sup_{x\in K} |\partial^\alpha \phi(x)|,$$

where K is any compact subset, makes C_0^∞ into a topological space, a Freche' space. We say that $\phi_n \to \phi$ in C_0^∞ if ϕ and all ϕ_j's are supported in a fixed compact set K and $\rho_{\alpha,K}(\phi_n - \phi) \to 0$, as $n \to \infty$ for every fixed α.
A linear map $L : C_0^\infty \to \mathbf{C}$ is called continuous if $\phi_n \to \phi$ implies that $L(\phi_n) \to L(\phi)$. If L is linear it is equivalent to only assume continuity at $\phi = 0$.

Definition. Let $\mathcal{D}'$ denote the dual space of C_0^∞, i.e. the space of all continuous linear functionals : $C_0^\infty \to \mathbf{C}$. $\mathcal{D}'$ is called the space of distributions.

The continuity is equivalent to that for every compact K there are C, N so that

$$|L(\phi)| \le C \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \phi(x)|, \qquad \text{if} \qquad \operatorname{supp}\phi \subset K.$$

If f is a distribution we will write $\langle f, \phi\rangle$ for what we just called $L(\phi)$.

A bounded function can be viewed as a distribution given by $\langle f, \phi\rangle = \int f\phi\, dx$. In fact, if supp $\phi \subset K$, then $|\int f\phi\, dx| \le \int_K |f|\, dx\, \sup_x |\phi(x)| \le C \sup_x |\phi(x)|$. Even if f is a distribution we will sometimes use $\int f\phi\, dx$ to denote $\langle f, \phi\rangle$, keeping in mind that it is to be interpreted as a linear functional and not an usual integral.

Moreover, any derivative of a function is a distribution even if the function is not differentiable in the usual sense. In fact, the main motivation to introduce distributions is to generalize the concept of derivative. We define the derivative by

$$\int (\partial^\alpha f)\, \phi\, dx = (-1)^{|\alpha|} \int f\, \partial^\alpha \phi\, dx$$

This defines a distribution that agrees with the usual derivative if f is smooth by integrating by parts.
Problem 5.3 Let $\psi(t) = e^{-1/t}$, when $t > 0$, and $\psi(t) = 0$, when $t \le 0$. Show that $\psi \in C^\infty(\mathbf{R})$. Let $\eta(x) = \psi(1 - |x|^2)$, where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that $\eta(x) \in C_0^\infty(\mathbf{R}^n)$.

The delta function. The simplest example of a distribution which is not a function is the "delta function" at a $\delta_a(x) = \delta(x-a)$ defined by

$$\int \phi(x)\, \delta_a(x)\, dx = \phi(a), \quad \phi \in C_0^\infty$$

It is a distribution since it satisfies $|\langle \delta_a, \phi\rangle| \le \sup_x |\phi(x)|$.
Physically one can think of the delta function as a point charge; $\delta_a(x) = \infty$ when $x = a$ and $\delta_a(x) = 0$ when $x \neq a$ in such a way that the total change is $\int \delta_a(x)\, dx = 1$. One can think of it as a limit of a the sequence $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$, with $\int \phi\, dx = 1$. A third interpretation in one variable is as the derivative of the step function: Define the Heavyside function H by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. Then in the sense of distributions $H'(x) = \delta(x)$. In fact

$$\int H'(x)\phi(x)\, dx = -\int H(x)\phi'(x)\, dx = -\int_0^\infty \phi'(x)\, dx = \phi(0), \quad \phi \in C_0^\infty.$$