Lecture 7: Fundamental solutions using distribution theory. The fundamental solution E of a partial differential operator $P(D) = \sum a_\alpha \partial^\alpha$ is defined by

$$P(D)E = \delta$$

Using the fundamental solution one can solve the equation

$$P(D)u = F,$$

In fact $u = E * F$ satisfies

$$P(D)(E * F) = (P(D)E) * F = \delta * F = F$$

That you can let the derivatives fall on either factor follows from writing out the convolution integral and differentiating below the integral sign.

Let us first derive the fundamental solution of $\triangle$, i.e. E such that $\triangle E = \delta$. Since $\triangle$ and $\delta(x)$ are invariant under rotations we expect $E(x)$ to be invariant, i.e. $E(x) = f(|x|)$, where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and f is a distribution.
If f is smooth when $x \neq 0$ we have

$$\triangle f(|x|) = \sum_j \partial_j \partial_j f(|x|) = \sum_j \partial_j \Big(f'(|x|) \frac{x_j}{|x|} \Big) = \sum_j f''(|x|) \frac{x_j^2}{|x|^2} + f'(|x|) \Big(\frac{1}{|x|} - \frac{x_j^2}{|x|^3} \Big) = 0$$

and hence

$$f''(r) + \frac{n-1}{r} f'(r) = 0, \qquad r = |x| \neq 0$$

which has the solution

$$f(r) = c_n r^{-n+2}, \quad n > 2, \qquad \text{and} \qquad f(r) = c_2 \ln r, \quad n = 2.$$

We claim that for the right choice of constant c_n, in the sense of distribution theory

$$\triangle E(x) = \delta(x), \qquad \text{if} \quad E(x) = f(|x|),$$

i.e. we need to prove that

$$\langle \triangle E, \phi \rangle = \langle E, \triangle \phi \rangle = \phi(0) = \langle \delta, \phi \rangle, \qquad \phi \in C_0^\infty.$$

Recall the divergence theorem (see e.g. Evans Appendix C)

$$\int_\Omega \partial_j u \, dx = \int_{\partial\Omega} u n_j dS$$

where n is the outward unit normal. If we apply this to uv in place of u we get the integration by parts formula

$$\int_\Omega v \partial_j u \, dx = - \int_\Omega u \partial_j v \, dx + \int_{\partial\Omega} uv n_j dS.$$

We have if $n > 2$:

$$\langle E, \triangle\phi\rangle = \int \frac{c_n}{|x|^{n-2}} \triangle\phi(x)\, dx = \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{c_n}{|x|^{n-2}} \sum_j \partial_j(\partial_j\phi(x))\, dx$$

$$= \lim_{\varepsilon\to 0} -\int_{|x|\geq\varepsilon} \sum_j \partial_j \frac{c_n}{|x|^{n-2}} \partial_j\phi(x))\, dx + \int_{|x|=\varepsilon} \frac{c_n}{|x|^{n-2}} \sum_j \frac{x_j}{|x|} \partial_j\phi(x)\, dS$$

$$= \lim_{\varepsilon\to 0} \sum_j \int_{|x|\geq\varepsilon} \partial_j^2 (\frac{c_n}{|x|^{n-2}}) \phi(x)\, dx + \int_{|x|=\varepsilon} \frac{c_n}{|x|^{n-2}} \frac{x_j}{|x|} \partial_j\phi(x)\, dS - \int_{|x|=\varepsilon} \partial_j (\frac{c_n}{|x|^{n-2}})\, \phi(x) \frac{x_j}{|x|}\, dS$$

Here the first term vanishes because $\triangle|x|^{-n+2} = 0$, when $x \neq 0$. The second term is bounded by $\sup_x |\partial\phi(x)| c_n \varepsilon^{-n+2} \int_{|x|=\varepsilon} dS \leq C\varepsilon \to 0$. The last term is

$$c_n \int_{|x|=\varepsilon} \frac{n-2}{|x|^{n-1}} \phi(x)\, dS = \frac{c_n(n-2)}{\varepsilon^{n-1}} \int_{|x|=\varepsilon} \phi(x) dS \to c_n(n-2)\phi(0) \int_{|x|=1} dS$$

Problem 7.1 Prove the Divergence theorem.

The fundamental solution for the wave equation

$$\Box E = \delta(t,x), \quad \Box = \partial_t^2 - \triangle, \qquad (t,x) \in \mathbf{R}^{1+n}$$

is not hard to derive from the symmetries as well. Since $\Box$ is invariant under Lorentz transformations we expect the fundamental solution $E(t,x)$ to be invariant under Lorentz transformations as well, which means that it should be of the form $E(t,x) = f(t^2 - |x|^2)$, where f is a distribution. Plugging this into the equation gives after some calculation

$$4\rho f''(\rho) + 2(1+n)f'(\rho) = 0, \quad \rho = t^2 - |x|^2 \tag{7.1}$$

when $(t,x) \neq (0,0)$. This has the solution

$$\begin{cases} f(\rho) = & c_1 H(\rho), \quad \text{if} \quad n=1 \\ f(\rho) = & c_2 H(\rho)\rho^{-1/2}, \quad \text{if} \quad n=2 \\ f(\rho) = & c_3 \delta(\rho), \quad \text{if} \quad n=3 \end{cases} \tag{7.2}$$

The constants can be calculated in the same way as we did for the fundamental solution of $\triangle$.

$$\begin{cases} E(t,x) = & c_1 H(t-|x|), \quad \text{if} \quad n=1 \\ E(t,x) = & c_2 H(t-|x|)(t^2-|x|^2)^{-1/2}, \quad \text{if} \quad n=2 \\ E(t,x) = & c_3 \delta(t^2-|x|^2)H(t), \quad \text{if} \quad n=3 \end{cases} \tag{7.3}$$

Problem 7.2: Show in each case that (7.2) is a solution of (7.1).
Problem 7.3: If $n=3$ prove that

$$\delta(t^2 - |x|^2)H(t) = \delta(t-|x|)/2|x| \tag{7.4}$$

Problem 7.4 Prove that $E(t,x)$ given above are fundamental solutions of $\Box$ and find the constants c_n.

Using the fundamental solution E for $\Box$ we can now solve the Cauchy problem

$$\Box u(t,x) = F$$
$$u(0,x) = f(x), \quad u_t(0,x) = g(x)$$

In fact let $u_0(t,x) = u(t,x)H(t)$ and $F_0(t,x) = F(t,x) = H(t)$ then

$$\Box u_0(t,x) = \Box u(t,x)H(t) = (\Box u(t,x))H(t) + 2u_t(t,x)\delta(t) + u(t,x)\delta'(t)$$
$$= F(t,x)H(t) + 2u_t(0,x)\delta(t) + u(0,x)\delta'(t) - u_t(0,x)\delta(t) = F_0(t,x) + g(x)\delta(t) + f(x)\delta'(t)$$

and hence

$$u_0(t,x) = E*\big(F_0(t,x) + g(x)\delta(t) + f(x)\delta'(t)\big) = E*F_0 + E*(g(x)\delta(t)) + \partial_t E*(f(x)\delta(t))$$

Let us now derive the solution formula if $n=3$ in which case $E(t,x) = \delta(t-|x|)/4\pi|x|$ and hence
Problem 7.5 Show that
(7.5)

$$E*F_0(t,x) = \int\int F_0(t-s,x-y)\,\delta(s-|y|)\frac{1}{4\pi|y|}dyds = \int_{|y|\le t} \frac{F_0(t-|y|,x-y)}{4\pi|y|}dy$$

and that

$$E*(g(x)\delta(t)) = t\int_{\omega\in S^2} \frac{g(x-t\omega)}{4\pi}dS(\omega)$$

where $dS(\omega)$ is the surface measure on the sphere S^2.