CHAPTER VI

NONLINEAR PERTURBATIONS OF THE WAVE EQUATION

6.1. Introduction. In this chapter we shall discuss the solution of a nonlinear Cauchy problem in $\mathbf{R}^{1+n}$,

$$F(u, u', u'') = 0; \quad u(0, \cdot) = u_0, \ \partial_t u(0, \cdot) = u_1; \tag{6.1.1}$$

where u_j are small and of compact support or at least decrease fast at infinity. The variables will usually be denoted by $t \in \mathbf{R}$ and $x \in \mathbf{R}^n$, but sometimes we write x_0 instead of t to obtain symmetrical notation, and $\vec{x} = (x_1, \dots, x_n)$ to avoid confusion. We shall assume that $u = 0$ is a solution of the equation $F(u, u', u'') = 0$ and that the linearization there is the wave equation. Thus

$$F(u, u', u'') = \Box u + f(u, u', u'')$$

where

$$\Box = \partial_t^2 - \Delta \tag{6.1.2}$$

is the wave operator, also called the d'Alembertian, and f vanishes of second order at 0. Additional conditions will be imposed on f in low dimensions.

As an orientation we shall first rephrase a simple special case of Theorem 4.3.1 as a result on the Cauchy problem (6.1.1) when $n = 1$. Thus consider the Cauchy problem

$$\sum_{j,k=0}^{1} g_{jk}(u')\partial_j \partial_k u = 0; \quad u(0, \cdot) = \varepsilon u_0, \ \partial_t u(0, \cdot) = \varepsilon u_1; \tag{6.1.3}$$

where $\sum_{j,k=0}^{1} g_{jk}(0)\partial_j\partial_k = \partial_0^2 - \partial_1^2$. With $U_1 = \partial_t u$, $U_2 = \partial_x u$, this is equivalent to a system

$$\partial_t U + a(U)\partial_x U = 0; \quad U_1 = \varepsilon u_1, \ U_2 = \varepsilon u_0' \text{ if } t = 0.$$

Here

$$a(U) = \begin{pmatrix} (g_{01} + g_{10})/g_{00} & g_{11}/g_{00} \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues are given by

$$\lambda^2 g_{00} - \lambda(g_{01} + g_{10}) + g_{11} = 0.$$

When $U = 0$ this reduces to $\lambda^2 = 1$, and the corresponding eigenvectors of $a(0)$ are $(1, -\lambda)$. The projections of (u_1, u_0') on these directions are $f_\lambda(1, -\lambda)$ where $2f_1 = u_1 - u_0'$ and $2f_{-1} = u_0' + u_1$. The differentials of the eigenvalues at 0 are given by

$$2\lambda \, d\lambda + \lambda^2 dg_{00} - \lambda d(g_{01} + g_{10}) + dg_{11} = 0,$$

that is,

$$2\langle d\lambda,(1,-\lambda)\rangle = \sum_{j,k,l=0}^{1} g_{jkl}(-\lambda)^{j+k+l+1}, \quad g_{jkl} = \partial g_{jk}(0)/\partial u_l'.$$

If T_ε is the lifespan of the classical solution of (6.1.3), with $u_j \in C_0^\infty(\mathbf{R})$, then Theorem 4.3.1 states that

$$\lim_{\varepsilon\to 0} 1/(\varepsilon T_\varepsilon) = \max_{\lambda=\pm 1} -\tfrac{1}{2}\sum_{j,k,l=0}^{1} g_{jkl}(-\lambda)^{j+k+l+1} f_\lambda'(y). \tag{6.1.4}$$

To interpret this result we observe that the solution of the unperturbed wave equation with Cauchy data (u_0,u_1) can be written in the form $F_1(x-t)+F_{-1}(-x-t)$ where F_1 is the wave moving to the right and F_{-1} is the wave moving to the left. To determine $F_{\pm 1}$ we have the equations

$$F_1(x)+F_{-1}(-x) = u_0(x),\ F_1'(x)+F_{-1}'(-x) = -u_1(x),$$

and after differentiation of the first formula we find that $F_\lambda'(\lambda x) = -f_\lambda(x)$ for $\lambda = \pm 1$. Hence we can rewrite (6.1.4) in the form

$$\lim_{\varepsilon\to 0} 1/(\varepsilon T_\varepsilon) = \max \tfrac{1}{2}\sum_{j,k,l=0}^{1} g_{jkl}\hat\lambda_j\hat\lambda_k\hat\lambda_l F_\lambda''(y) \tag{6.1.5}$$

where $\hat\lambda = (-1,\lambda)$ and the maximum is taken with respect to $y\in\mathbf{R}$ and $\lambda = \pm 1$.

Our aim will be to prove results similar to (6.1.5) when $n = 2$ or $n = 3$, with T_ε replaced by $\sqrt{T_\varepsilon}$ and $\log T_\varepsilon$ respectively while there will be global existence theorems for $n > 3$. The reason for the improved behavior when the dimension n gets large is that solutions of the linear Cauchy problem in $\mathbf{R}^{1+n}$ with smooth data of compact support can be estimated by $(1+t)^{(1-n)/2}$, and that for the solution of the ordinary differential equation $du(t)/dt = \varepsilon(1+t)^{(1-n)/2}u(t)$ we have

$$\begin{aligned}
u(t) &= u(0)\exp \varepsilon t, && \text{if } n = 1;\\
u(t) &= u(0)\exp\left(2\varepsilon(\sqrt{1+t}-1)\right), && \text{if } n = 2;\\
u(t) &= u(0)(1+t)^\varepsilon, && \text{if } n = 3;\\
u(t) &\le |u(0)|e^{2\varepsilon}, && \text{if } n > 3.
\end{aligned}$$

In view of the crucial role of the linear wave equation we shall start with a thorough study of its solutions in Section 6.2. This can be based on Fourier transforms or the completely explicit fundamental solution. The Friedlander radiation field which describes the asymptotic behavior of the solutions is studied in Section 6.2, first using the fundamental solution of $\Box$, and then using a conformal compactification. However, for equations with variable coefficients or nonlinear equations we must use the energy integral method to derive estimates. It is presented in Section 6.3 where we also outline a proof of existence and uniqueness of solutions to the Cauchy problem for linear hyperbolic equations with variable coefficients. In Section 6.4 we then discuss briefly some "interpolation inequalities" and Sobolev inequalities which are indispensible in nonlinear problems, and we show how they work in a simple case by proving local existence theorems for the nonlinear Cauchy problem (6.1.1). Global existence theorems are proved in Section 6.5 using an idea of Klainerman [3] which exploits the infinitesimal generators of the Lorentz group, that is, vector fields commuting with $\Box$. In dimensions 2 and 3 the results only give at first the order of magnitude of a finite lower bound for the lifespan in terms of the size of the initial data, but they are then improved to asymptotic lower bounds for the lifespan similar to (6.1.5). It is plausible that these really give the asymptotic behavior as $\varepsilon \to 0$ of the lifespan but this is only known from work of F. John [2] in some special cases closely related to (6.1.5). (See also the remarks at the end of Section 6.5.) In Section 6.6 the results of Christodoulou [1] and Klainerman [4] on existence of solutions for all $t \geq 0$ when $n = 3$ and the so called null condition is fulfilled are proved with the methods of Klainerman. Section 6.7 gives the proof of Christodoulou based on a conformal compactification, which explains the role of the null condition better.

6.2. The linear wave equation. In this section we shall discuss the behavior at infinity of the solution of the wave equation in $\mathbf{R}^{1+n}$, $n \geq 1$,

$$\Box u = (\partial_t^2 - \Delta)u = 0, \tag{6.2.1}$$

with Cauchy data

$$u = f,\ \partial_t u = g \quad \text{when } t = 0, \tag{6.2.2}$$

where $f, g \in C_0^\infty(\mathbf{R}^n)$. At first we assume that $f = 0$, which implies that in the sense of distribution theory

$$u(t,x) = \int E(t, x-y) g(y)\, dy, \quad t > 0,$$

where E is the fundamental solution (see e.g. Hörmander [4, Section 6.2])

$$E = \tfrac{1}{2}\pi^{\frac{1-n}{2}} \chi_+^{\frac{1-n}{2}}(t^2 - |x|^2).$$

Here $x = (x_1, \dots, x_n)$ and

$$\chi_+^a(s) = s^a/\Gamma(a+1), \quad s > 0, \quad \chi_+^a(s) = 0, \quad s \leq 0, \quad \text{if } \operatorname{Re} a > -1,$$
$$d\chi_+^a/ds = \chi_+^{a-1} \quad \text{for all } a \in \mathbf{C}.$$

Thus $\chi_+^{-k} = \delta_0^{(k-1)}$, $k = 1, 2, \dots$, is supported by the origin.

Set $x = r\omega$ where $r = |x|$ and $\omega \in S^{n-1}$. Then $r \leq t + M$ in $\operatorname{supp} u$ if $|y| \leq M$ in $\operatorname{supp} g$, for $|x| \leq |x-y| + |y| \leq t + M$ in the support of the "integrand". When n is odd we also have $r \geq t - M$, for $|x| \geq |x-y| - |y| \geq t - M$ (Huygens' principle). When n is even we note instead that $u(t,x) = O(t^{1-n})$ if $|x| < t/2$ and $t \to \infty$, for E is homogeneous of degree $1-n$. Differentiation gives a faster decrease, $\partial^\alpha u(t,x) = O(t^{1-n-|\alpha|})$ when $|x| < t/2$. The main contributions to u must therefore always occur when $r - t$ is small compared to t, so we set $r = t + \varrho$ where $-r \leq \varrho \leq M$, if $2r \geq t \geq r - M$. Then

$$t^2 - |x-y|^2 = (r-\varrho)^2 - |r\omega - y|^2 = 2r(\langle \omega, y\rangle - \varrho) + \varrho^2 - |y|^2,$$

and we obtain by the homogeneity of E

$$\begin{aligned}(2\pi r)^{\frac{n-1}{2}} u(t,x) &= \tfrac{1}{2}\int \chi_+^{\frac{1-n}{2}}(\langle \omega, y\rangle - \varrho + (\varrho^2 - |y|^2)/2r) g(y)\, dy \\ &= \tfrac{1}{2}\int \chi_+^{\frac{1-n}{2}}(s - \varrho + \varrho^2/2r) G(\omega, r^{-1}, s)\, ds \\ &= \tfrac{1}{2}\int \chi_+^{\frac{1-n}{2}}(s + (t^2 - r^2)/2r) G(\omega, r^{-1}, s)\, ds.\end{aligned}$$

Here

$$G(\omega, z, s) = \int \delta(s - \langle \omega, y\rangle + |y|^2 z/2) g(y)\, dy$$

is a C^∞ function in $S^{n-1} \times [0, 1/2M] \times \mathbf{R}$ with $|s| \le 5M/4$ in the support and $G(\omega, 0, s) = R(\omega, s; g)$, where

$$R(\omega, s; g) = \int \delta(s - \langle \omega, y\rangle) g(y)\, dy = \int_{\langle \omega, y\rangle = s} g(y)\, dS(y) \tag{6.2.3}$$

is the *Radon transform* of g, which has support in $S^{n-1} \times [-M, M]$. Hence

$$r^{\frac{n-1}{2}} u(t,x) = F(\omega, r^{-1}, \varrho) \tag{6.2.4}$$

where the convolution

$$F(\omega, z, \varrho) = \tfrac{1}{2}(2\pi)^{\frac{1-n}{2}} \int \chi_+^{\frac{1-n}{2}}(s - \varrho + \varrho^2 z/2) G(\omega, z, s) ds \tag{6.2.5}$$

is a C^∞ function in $S^{n-1} \times [0, 1/2M] \times \mathbf{R}$ with $\varrho \le M$ in the support. This result is due to Friedlander [1, 2] who only assumed that u satisfies the wave equation for large $|x|$. The restriction F_0 of F to $z = 0$ is the *Friedlander radiation field,*

$$F_0(\omega, \varrho) = \tfrac{1}{2}(2\pi)^{\frac{1-n}{2}} \int \chi_+^{\frac{1-n}{2}}(s - \varrho) R(\omega, s; g)\, ds.$$

From the homogeneity of $\chi_+^{\frac{1-n}{2}}$ it follows at once that $F_0(\omega, \varrho)$ is a polyhomogeneous symbol in ϱ of degree $(1-n)/2$ (cf. Hörmander [4, Def. 18.1.5]). We have, still with $\varrho = r - t$ and $x = r\omega$,

$$|u(t,x) - r^{\frac{1-n}{2}} F_0(\omega, \varrho)| \le C((1+\varrho)/r)(r(1+\varrho))^{\frac{1-n}{2}}, \quad \text{if } r > t/2 > 1. \tag{6.2.6}$$

This follows at once from the differentiability of F when $\varrho \ge -2M$, say. When n is even we must also use that for $\varrho < -2M$ we have

$$|\partial F(\omega, z, \varrho)/\partial z| \le C(1 + |\varrho|)^{\frac{3-n}{2}},$$

since differentiation of (6.2.5) shows that $\partial F/\partial z$ can be estimated by a constant times $\varrho^2(1+|\varrho|)^{-\frac{1+n}{2}} + (1+|\varrho|)^{\frac{1-n}{2}}$.

So far we have only studied the solution of (6.2.1), (6.2.2) when $f = 0$. However, an approximating radiation field always exists. In fact, choosing $\psi \in C_0^\infty(\mathbf{R})$ equal to 1 in $[1, \infty)$ and 0 in $(-\infty, 0)$ we obtain when $t > 1$

$$u = E * K, \quad K = \Box(\psi(t)u) = 2\psi'(t)\partial u/\partial t + \psi''(t)u.$$

Thus $K \in C_0^\infty(\mathbf{R}^{1+n})$ if $f, g \in C_0^\infty(\mathbf{R}^n)$, and in the sense of distribution theory

$$u(t,x) = \int_0^1 ds \int E(t-s, x-y) K(s,y) dy$$

is a superposition of solutions of the form discussed already. If (6.2.6) is valid for a solution u of (6.2.1) then (6.2.6) remains valid if u is replaced by $u(\cdot - s, \cdot)$ and F_0 is replaced by $F_0(\cdot, \cdot + s)$. In view of the uniformity in g of the estimate (6.2.6) in the case where we have proved it, we conclude that there is always some F_0, obviously uniquely defined, such that (6.2.6) holds and F_0 is a symbol in ϱ of order $(1-n)/2$, which is smooth in ω. The radiation field depends continuously on the initial data f, g. If F_0 is the radiation field of u we have just observed that the radiation field of $(u(\cdot+s,\cdot)-u)/s$ is $(F_0(\cdot,\cdot-s)-F_0)/s$, so we conclude that the radiation field of $\partial u/\partial t$ is $-\partial F_0/\partial\varrho$. If u satisfies (6.2.1), (6.2.2) with $f=0$, then $\partial u/\partial t$ satisfies (6.2.1) and (6.2.2) with f replaced by g and g replaced by 0. It follows that the radiation field is given in general by

$$F_0(\omega,\cdot)=\tfrac{1}{2}(2\pi)^{\frac{1-n}{2}}\chi_-^{\frac{1-n}{2}}*(R(\omega,\cdot;g)-R(\omega,\cdot;f)'),\quad \chi_-^{\frac{1-n}{2}}=\chi_+^{\frac{1-n}{2}}(-\cdot), \tag{6.2.7}$$

with convolution and differentiation taken in the variable ϱ indicated by a dot.

By (6.2.6) we have for bounded ϱ, small h and large r

$$u(t,x+h)=r^{\frac{1-n}{2}}(F_0(\omega,\varrho)+\partial F_0(\omega,\varrho)/\partial\varrho\langle\varrho',h\rangle+O(|h|^2+1/r)),$$

so the radiation field of $u(t,x+h)$ is $F_0(\omega,\varrho)+\partial F_0(\omega,\varrho)/\partial\varrho\langle\omega,h\rangle+O(|h|^2)$. In view of the continuous dependence just pointed out it follows that the radiation field of $\partial u/\partial x_j$ is $\omega_j\partial F_0/\partial\varrho$; the other terms obtained by formal differentiation of (6.2.6) are absorbed by the error term. However, to give a precise description of the behavior of u at infinity we must also apply other differential operators which exploit the invariance of the wave operator under the Lorentz group and homotheties. These are the vector fields

$$Z_{jk}=\lambda_j x_j\partial/\partial x_k-\lambda_k x_k\partial/\partial x_j,\quad j,k=0,\dots,n, \tag{6.2.8}$$

where $\lambda=(1,-1,\dots,-1)$, which all commute with $\Box$, and the radial vector field

$$Z_0=\sum_0^n x_j\partial/\partial x_j, \tag{6.2.9}$$

for which $[\Box,Z_0]=\Box Z_0-Z_0\Box=2\Box$. For an arbitrary product Z^I of such vector fields it is clear that $\Box Z^I u=0$ if $\Box u=0$. We have

$$Z_{jk}(|x|-t)=0,\quad j,k\neq 0;\quad Z_{0k}(|x|-t)=(t-|x|)x_k/|x|,\quad k\neq 0;\quad Z_0(|x|-t)=|x|-t.$$

Thus

$$Z^I(|x|^{\frac{1-n}{2}}F_0(x/|x|,|x|-t))=|x|^{\frac{1-n}{2}}F_I(x/|x|,|x|-t),$$

where $F_I(\omega,\varrho)$ is also a polyhomogeneous symbol of order $(1-n)/2$ in ϱ. In fact, when $Z=Z_{jk}$ or Z_0 operates on a homogeneous function it gives another one of the same degree, and when it operates on $\varrho=|x|-t$ we have just seen that Z acts as the operator $\varrho\partial/\partial\varrho$ followed by multiplication with a homogeneous function of x of degree 0. If $s\mapsto T(s)$ is the one parameter group of linear transformations generated by Z, then we deduce as above from (6.2.6) that the radiation field of $u\circ T(s)$ is $r^{\frac{n-1}{2}}(1+sZ)r^{\frac{1-n}{2}}F_0(\omega,\varrho)+O(s^2)$, so the radiation field of Zu is $r^{\frac{n-1}{2}}Zr^{\frac{1-n}{2}}F_0(\omega,\varrho)$. Repeating the argument we find that (6.2.6) implies the following result:

Theorem 6.2.1. *If u is a solution of the Cauchy problem (6.2.1), (6.2.2) with $f,g\in C_0^\infty(\mathbf{R}^n)$, then*

$$|\partial^\alpha Z^I u|\le C_{\alpha,I}(1+|t|+|t^2-|x|^2|)^{\frac{1-n}{2}} \tag{6.2.10}$$

for arbitrary α and I. More precisely, if F_0 is the radiation field of u, then

$$|\partial^\alpha Z^I(u(t,x)-r^{\frac{1-n}{2}}F_0(\omega,\varrho))|\le C(1+\varrho)^{\frac{3-n}{2}}t^{-\frac{(1+n)}{2}},\quad \textit{if } r>t/2>1. \tag{6.2.11}$$

When $n=1$ then (6.2.6) simplifies to

$$u(t,x)=F(\operatorname{sgn}x,|x|-t)$$

for large t when $x\neq 0$, so $F(\omega,\cdot)$ is the function F_ω of the introduction for $\omega=\pm1$. When $n=2,3$ we shall obtain results similar to (6.1.5) for the lifespan of the solution of a nonlinear Cauchy problem where F_λ is replaced by the radiation field. The following theorem will be important in the interpretation of the result:

Theorem 6.2.2. *If $f, g \in C_0^\infty(\mathbf{R}^n)$ then the radiation field F_0 of the solution u of (6.2.1), (6.2.2) is not identically 0 unless f and g are identically 0.*

Proof. If F_0 is identically 0 it follows from the theorem of supports and (6.2.7) that

$$R(\omega, \varrho; g) - dR(\omega, \varrho; f)/d\varrho \equiv 0.$$

Since $R(\omega, \varrho; \cdot) = R(-\omega, -\varrho; \cdot)$, this is equivalent to

$$R(-\omega, -\varrho; g) - dR(-\omega, -\varrho; f)/d\varrho \equiv 0.$$

If we carry out the differentiation and replace $(-\omega, -\varrho)$ by (ω, ϱ) afterwards, it follows that

$$R(\omega, \varrho; g) + dR(\omega, \varrho; f)/d\varrho \equiv 0,$$

so $R(\omega, \varrho; g) \equiv 0$ and $dR(\omega, \varrho; f)/d\varrho \equiv 0$, which implies $R(\omega, \varrho; f) \equiv 0$. Hence f and g vanish identically.

Remark. The proof shows that the projection of the support of $R(\omega, \varrho; g) - R(\omega, \varrho; f)'$ on S^{n-1} cannot omit two antipodal points. In fact, if $R(\omega, \varrho; g)$ vanishes for all ω in an open set, then it vanishes identically since the Fourier transform of g will vanish in the open cone which it generates.

When $n = 3$ and f, g are functions of $r = |x|$ only, hence C^∞ functions of r^2, then the solution of (6.2.1), (6.2.2) is a function of t and r, and the equation (6.2.1) can be written

$$(ru)''_{tt} - (ru)''_{rr} = 0.$$

Hence

$$u(t, x) = r^{-1} F_0(r - t), \quad t > M, \ r = |x|,$$

so the left-hand side of (6.2.6) vanishes for large t. It is easy to compute F_0, for

$$R(\omega, \varrho; g) = \int_{|\varrho|}^\infty g(t) d\pi(t^2 - \varrho^2) = 2\pi \int_\varrho^\infty t g(t)\, dt,$$

and similarly for $R(\omega, \varrho; f)$. It follows that the radiation field, which only depends on ϱ, is given by

$$dF_0(\varrho)/d\varrho = (d(\varrho f(\varrho))/d\varrho - \varrho g(\varrho))/2. \tag{6.2.12}$$

The arguments in this section have all been based on the properties of the fundamental solution of $\Box$, essentially as in Friedlander [1]. We shall now discuss another approach from Friedlander [3] which exploits the conformal map from the Minkowski space M^{1+n}, that is, $\mathbf{R}^{1+n}$ with the standard Lorentz metric, to the Einstein universe $\mathbf{R} \times S^n$, defined in Section A.4 of the appendix. The scalar curvature of the Einstein universe is minus that of S^n with the standard metric, so it is $-n(n-1)$ by (A.2.10). The conformal d'Alembertian (see Section A.3) is therefore

$$\partial_T^2 - \widetilde{\Delta} + n(n-1)(n-1)/4n = \widetilde{\Box} + (n-1)^2/4$$

where $\widetilde{\Delta}$ is the Laplace operator on S^n and $\widetilde{\Box}$ is the d'Alembertian on $\mathbf{R} \times S^n$. If u is a solution of $\Box u = 0$ in M^{1+n}, and with the notation in Section A.4

$$\tilde{u}(T, X) = (\cos T + X_0)^{(1-n)/2} u(\Psi(T, X)), \quad \text{when } \cos T + X_0 > 0,\ 0 \le T < \pi, \tag{6.2.13}$$

it follows that $(\tilde{\Box}+(n-1)^2/4)\tilde{u}=0$ and conversely. If the Cauchy data of u are in $\mathcal{S}(\mathbf{R}^n)$, then those of $\tilde{u}$ are in $C^\infty(S^n)$, and they vanish of infinite order at the pole corresponding to infinity in $\mathbf{R}^n$. We shall prove in Section 6.3 that this linear Cauchy problem has a solution $\tilde{u}\in C^\infty(\mathbf{R}\times S^n)$, and below we shall also outline a proof using an explicit fundamental solution of $\tilde{\Box}+(n-1)^2/4$. Since

$$\cos T+X_0=\Omega=2\big((1+(|x|-t)^2)(1+(|x|+t)^2)\big)^{-\frac{1}{2}},$$

it follows at once that

$$|u(t,x)|\le C\big((1+(|x|-t)^2)(1+(|x|+t)^2)\big)^{\frac{1-n}{2}},$$

which is the estimate (6.2.10) (without differentiations). To conclude (6.2.11) we must determine the limit of $\Psi^{-1}(t,(t+\varrho)\omega)$, $\omega\in S^{n-1}$, as $t\to+\infty$. The corresponding coordinates T,α are defined by

$$\sin T=\Omega t,\ \sin\alpha=\Omega(t+\varrho),\ \cos\alpha=\tfrac{1}{2}\Omega(1-\varrho(t+r)),$$

and since $\Omega\sim t^{-1}(1+\varrho^2)^{-\frac{1}{2}}$, it follows that

$$\sin\alpha\to(1+\varrho^2)^{-\frac{1}{2}},\quad \cos\alpha\to-\varrho(1+\varrho^2)^{-\frac{1}{2}},\quad \sin T\to(1+\varrho^2)^{-\frac{1}{2}}.$$

We have $T+\alpha\to\pi$ since $T>0$ and $\Psi(T,X)$ remains finite when $|T|+\alpha$ has a bound $<\pi$. Hence

$$\text{(6.2.14)}\quad t^{\frac{n-1}{2}}(1+\varrho^2)^{\frac{n-1}{4}}u(t,(t+\varrho)\omega)\to\tilde{u}(\pi-\alpha,(\cos\alpha,\sin\alpha\omega));\quad 0<\alpha<\pi,\cot\alpha=-\varrho,$$

which means that

$$\text{(6.2.15)}\qquad F_0(\omega,\varrho)(1+\varrho^2)^{\frac{n-1}{4}}=\tilde{u}(\pi-\alpha,(\cos\alpha,\sin\alpha\,\omega)).$$

For bounded ϱ the difference between (T,α) and the limit is $O(1/t)$, so the difference between the two sides in (6.2.14) is actually $O(1/t)$, which gives (6.2.11) when no derivatives are present. What remains is to examine what application of the differential operators Z and ∂ mean in the Einstein model; the full result (6.2.11) will then be a consequence of the fact that $\tilde{u}\in C^\infty$, with vanishing of infinite order at the infinitely distant point $T=0$, $\alpha=\pi$. We leave this for the interested reader but will return to the conformal d'Alembertian in Section 6.7.

It would not have been necessary to use the results of Section 6.3 here, for we can write down the fundamental solution for the conformal d'Alembertian in $\mathbf{R}\times S^n$ explicitly. To do so we first take the pole at $T=0$ and the point in S^n corresponding to $\alpha=0$. Recall that the fundamental solution of $\Box$ in M^{1+n} is for $t>0$

$$E=\tfrac{1}{2}\pi^{\frac{1-n}{2}}\chi_+^{\frac{1-n}{2}}(t^2-|x|^2).$$

With the notation in (A.4.2) we set $t=\Omega^{-1}\sin T$, $x=\Omega^{-1}X$, $\Omega=\cos T+\cos\alpha$, and note that

$$t^2-|x|^2=(\sin^2T-\sin^2\alpha)/\Omega^2=(\cos\alpha-\cos T)/\Omega.$$

Hence $E(\Psi(T,X))=\frac{1}{2}\pi^{(1-n)/2}\Omega^{(n-1)/2}\chi_+^{(1-n)/2}(\cos\alpha-\cos T)$. Now

$$(\tilde{\Box}+(n-1)^2/4)(\Omega^{\frac{1-n}{2}}E(\Psi(T,X))=\Omega^{-\frac{n+3}{2}}\Psi^*\Box E=2^{\frac{n-1}{2}}\delta_0$$

where δ_0 should be replaced by $\delta_0/\sqrt{g}$ if the coordinates in the Einstein universe are not chosen geodesic. (Recall that δ_0 is a distribution density.) Hence

$$(6.2.16)\qquad (\widetilde{\Box}+(n-1)^2/4)\widetilde{E}=\delta_0,\quad \text{if } \widetilde{E}=\tfrac{1}{2}(2\pi)^{\frac{1-n}{2}}\chi_+^{\frac{1-n}{2}}(\cos\alpha-\cos T),$$

provided that $\alpha+|T|<\pi$ and we define $\widetilde{E}(T,\cdot)=0$ when $T\le 0$. (6.2.16) remains true for $T<\pi$. If n is odd then $\chi_+^{(1-n)/2}$ is even or odd, so replacing T by $\pi-T$ and α by $\pi-\alpha$ shows that we have a solution of the homogeneous conformal d'Alembertian when $\pi<\alpha+T$ and $T<\pi$, and the solution has the same distribution limits on the characteristic surface $\alpha+T=\pi$ from both sides. If n is even, then we obtain by this substitution the distribution $\frac{1}{2}(2\pi)^{(1-n)/2}\chi_-^{(1-n)/2}(\cos\alpha-\cos T)$, which satisfies the homogeneous conformal d'Alembertian for $\alpha+|T|<\pi$ since $\Box\chi_-^{(1-n)/2}(t^2-|x|^2)=0$ in $\mathbf{R}^{1+n}$ for even n. (See Hörmander [4, Theorem 6.2.1].) Again we conclude that (6.2.16) is a fundamental solution for $T<\pi$.

If the pole on S^n defined by $\alpha=0$ is replaced by another point, we just have to replace α in (6.2.16) by the geodesic distance s to that point along S^n. We can therefore write down the fundamental solution for $T<\pi$ with pole at an arbitrary point in S^n. This suffices to conclude that the Cauchy problem for the conformal d'Alembertian with C^∞ initial data when $T=0$ has a solution in C^∞ for $T<\pi$, and iteration of this conclusion proves that there is a C^∞ solution for all $T\ge 0$.

It is in fact easy to obtain a global fundamental solution. Assume first that *n is odd.* Then the fundamental solution arrives as $T\to\pi-0$ as $(-1)^{(3-n)/2}$ times the backward fundamental solution at the antipode, and it must be continued as $(-1)^{(1-n)/2}$ times the forward fundamental solution with pole at the antipode and time π. At time 2π it arrives as minus the backward fundamental solution at the original point, so the fundamental solution then repeats with period 2π in T. Assume now that *n is even.* Then the fundamental solution continues beyond the antipodal point at time π with support outside the characteristic conoid there, and it arrives at time 2π as the backward fundamental solution at the original point in S^n. It is then continued with a change of sign to the next interval $2\pi<T<4\pi$ and so on, with a period of 4π. The fundamental solution is for every n a continuous function of T with values in $\mathcal{D}'(S^n)$ for all T, and when $T\ne 0$ it is infinitely differentiable with values in $\mathcal{D}'$.

6.3. The energy integral method. The basic energy estimate for the d'Alembertian $\Box$ is obtained from the identity

$$(6.3.1)\qquad 2\partial_0 u\Box u=\partial_0|u'|^2-2\sum_1^n\partial_j(\partial_0 u\partial_j u),$$

where

$$|u'|^2=\sum_0^n|\partial_j u|^2,\quad \partial_j=\partial/\partial x_j.$$

If $u\in C^2$ and u vanishes for large $\vec{x}=(x_1,\dots,x_n)$, then integration with respect to $\vec{x}$ gives

$$\partial_0\|u'(x_0,\cdot)\|^2=2\int\partial_0 u\Box u\,d\vec{x}\le 2\|u'(x_0,\cdot)\|\|\Box u(x_0,\cdot)\|,$$

where the norms are L^2 norms with respect to $\vec{x}$. Thus

$$\partial_0\|u'(x_0,\cdot)\|\le\|\Box u(x_0,\cdot)\|,$$

which gives after integration

$$\|u'(x_0,\cdot)\| \le \|u'(0,\cdot)\| + \int_0^{x_0} \|\Box u(t,\cdot)\| dt. \tag{6.3.2}$$

In particular it follows that $u \equiv 0$ if $\Box u = 0$ and the Cauchy data vanish, which also follows at once by taking Fourier transforms in $\vec{x}$.

The strength of the energy integral method is its stability under perturbations of the equation whereas methods based on the Fourier transformation break down at once. As a first step towards more general energy identities we consider a hyperbolic operator $\sum_{j,k=0}^n g^{jk}\partial_j\partial_k$ with a constant symmetric matrix (g^{jk}). If K^i are constants then (6.3.1) generalizes to

$$2\sum_{i=0}^n K^i\partial_i u \sum_{j,k=0}^n g^{jk}\partial_j\partial_k u = \sum_{i,j=0}^n \partial_j(T_i^j(u)K^i), \tag{6.3.3}$$

$$T_i^j(u) = 2\sum_{k=0}^n g^{jk}\partial_k u\partial_i u - \delta_i^j \sum_{k,l=0}^n g^{kl}\partial_k u\partial_l u. \tag{6.3.4}$$

The definition of T_i^j means that if ν is a covector, then

$$\sum_{i,j=0}^n T_i^j(u)K^i\nu_j = 2\langle K,u'\rangle\langle gu',\nu\rangle - \langle K,\nu\rangle\langle gu',u'\rangle. \tag{6.3.4$'$}$$

If A denotes the bilinear form defined by the inverse of g and $N = g\nu$, $U = gu'$ are the vectors corresponding to the covectors ν, u', then the *energy* on a surface with conormal ν, that is, the integrand obtained there by integrating out the right-hand side of (6.3.3), becomes

$$\begin{aligned}\sum_{i,j=0}^n T_i^j(u)K^i\nu_j &= 2A(U,N)A(U,K) - A(U,U)A(N,K)\\ &= A(2UA(U,K) - KA(U,U),N).\end{aligned} \tag{6.3.5}$$

Lemma 6.3.1. (6.3.5) *is positive definite in* U *if* A *has Lorentz signature and*

$$A(K,K) > 0, \quad A(N,N) > 0, \quad A(N,K) > 0,$$

that is, if K *and* N *are in the same open Lorentz half cone.*

Proof. Set $V = 2UA(K,U) - KA(U,U)$. Then

$$A(V,V) = A(K,K)A(U,U)^2 \ge 0, \quad A(V,K) = A(K,U)^2 + A(K,U)^2 - A(K,K)A(U,U),$$

where the difference of the last two terms is positive if $U \notin \mathbf{R}K$, by the reversed Cauchy-Schwarz inequality which follows from the Lorentz signature. Hence $A(V,K) > 0$ and $A(V,V) \ge 0$, which implies $A(V,N) > 0$.

One can also prove the lemma by an easy computation. If K is normalized with $A(K,K) = 1$, we can choose coordinates so that $A(x) = x_0^2 - x_1^2 - \cdots - x_n^2$, $K = (1,0,\dots,0)$, and obtain

$$V = (U_0^2 + r^2, 2U_0 r\omega) \quad \text{if } (U_1,\dots,U_n) = r\omega,\ |\omega| = 1.$$

We have

$$(U_0^2 + r^2, 2U_0 r) = (U_0^2 + r^2)(1, \sin 2\theta),$$

if θ is the polar angle in a Euclidean (U_0, r) plane. If $N = (N_0, \vec{N})$ with $|\vec{N}| < N_0$ then

$$A(V,N) = (U_0^2 + r^2)N_0 - 2U_0 r\langle\omega,\vec{N}\rangle \ge (N_0 - |\vec{N}|)(U_0^2 + r^2).$$

If g and K are allowed to depend on x, we get additional terms in (6.3.3) compensating those where ∂_j acts on the coefficients of T_i^j. The following simple but useful proposition shows that these can often be taken care of:

Proposition 6.3.2. *Let $u \in C^2$ satisfy a differential equation*

$$\Box u + \sum_{j,k=0}^{n} \gamma^{jk}(x)\partial_j\partial_k u = f, \quad 0 \le x_0 < T,$$

and assume that $u = 0$ for large $\vec{x}$. If

$$|\gamma| = \sum_{j,k=0}^{n} |\gamma^{jk}| \le \tfrac{1}{2}, \quad 0 \le x_0 < T,$$

it follows for $0 \le x_0 < T$ that

$$\|u'(x_0,\cdot)\| \le 2\Big(\|u'(0,\cdot)\| + \int_0^{x_0} \|f(t,\cdot)\|\,dt\Big)\exp\Big(\int_0^{x_0} 2|\gamma'(t)|\,dt\Big), \tag{6.3.6}$$

where the norms are L^2 norms with respect to $\vec{x}$ and

$$|\gamma'(t)| = \sum_{i,j,k=0}^{n} \sup|\partial_i\gamma^{jk}(t,\cdot)|.$$

Proof. We shall use (6.3.3) with the modification just indicated, with g^{jk} equal to γ^{jk} plus the coefficients of $\Box$, and $K = (1,0,\dots,0)$. Then

$$T_0^0(x,u) = |u'|^2 + \gamma^{00}(\partial_0 u)^2 - \sum_{k,l=1}^{n} \gamma^{kl}\partial_k u\partial_l u \ge |u'|^2/2.$$

With

$$\begin{aligned} R(x,u) &= 2\sum_{j,k=0}^{n} \partial_j\gamma^{jk}\partial_k u\partial_0 u - \sum_{k,l=0}^{n} \partial_0\gamma^{kl}\partial_k u\partial_l u \\ &= \partial_0\gamma^{00}(\partial_0 u)^2 - \sum_{k,l=1}^{n} \partial_0\gamma^{kl}\partial_k u\partial_l u + 2\sum_{j=1}^{n}\sum_{k=0}^{n} \partial_j\gamma^{jk}\partial_k u\partial_0 u \end{aligned}$$

we obtain

$$\partial_0 \int T_0^0(x,u)d\vec{x} \le 2\|f(x_0,\cdot)\|\|\partial_0 u(x_0,\cdot)\| + \int R(x,u)\,d\vec{x},$$

hence

$$\partial_0 E(x_0)^2 \le 2\sqrt{2}\|f(x_0,\cdot)\|E(x_0) + 4|\gamma'(x_0)|E(x_0)^2,$$

if $E(x_0)^2 = \int T_0^0(x,u)\,d\vec{x}$. Thus

$$\partial_0 E(x_0) \le \sqrt{2}\|f(x_0,\cdot)\| + 2|\gamma'(x_0)|E(x_0).$$

If we multiply by the integrating factor $\exp(-\int_0^{x_0} 2|\gamma'(t)|\,dt)$ and integrate, the estimate (6.3.6) follows.

The estimate (6.3.6) suffices to prove most of the existence theorems for nonlinear perturbations of $\Box$ in this chapter. However, it has the weakness that no estimate of u itself is obtained, and for the proof in Section 6.6 of some more refined results when $n = 3$

one also needs a more sophisticated estimate which we shall now discuss. (Alternative proofs in Section 6.7 do not require this estimate.)

Let us first assume that g^{jk} are constant and look for the variable vector fields K such that for some variable scalar L

$$(6.3.3)' \quad \begin{aligned} &2(\sum_i K^i\partial_i u + Lu)\sum_{j,k} g^{jk}\partial_j\partial_k u = \sum_j \partial_j(\sum_i T_i^j(u)K^i + 2Lu\sum_k g^{jk}\partial_k u) \\ &-2\sum_{i,j,k}\partial_j K^i g^{jk}\partial_k u\partial_i u + (\sum_i \partial_i K^i - 2L)\sum_{j,k} g^{jk}\partial_j u\partial_k u - 2\sum_{j,k} ug^{jk}\partial_j L\partial_k u \end{aligned}$$

is an exact divergence. With the notation $K' = (\partial_j K^i)$ we see that cancellation requires that

$$K'g + g\,{}^tK' = (\sum_i \partial_i K^i - 2L)g,$$

which means that K is a conformal vector field with respect to the metric defined by the dual quadratic form A. When $n \geq 2$ there are not many:

Proposition 6.3.3. *If $n \geq 2$ then all smooth vector fields K such that*

$$(6.3.7) \qquad K'g + g\,{}^tK' = 2Fg$$

are of the form

$$(6.3.8) \qquad K = 2A(x,\theta)x - A(x,x)\theta + K_0$$

where θ is a constant vector, $F(x) = 2A(x,\theta) + c$ with a constant c, and K_0 is the sum of $c\sum_0^n x_j\partial_j$, a constant vector field, and a linear combination of the vector fields

$$(6.3.9) \qquad (\partial_j A(x))\partial_k - (\partial_k A(x))\partial_j; \quad j,k = 0,\dots,n.$$

Proof. We may assume that g is diagonal and write g^i instead of g^{ii}. Then (6.3.7) can be written

$$\partial_j K^i g^j + g^i\partial_i K^j = 0, \quad i \neq j; \quad \partial_i K^i = F.$$

If i,j,k are different indices (recall that we have assumed that the dimension $n+1$ is at least 3), we obtain

$$\partial_k\partial_j K^i/g^i = -\partial_k\partial_i K^j/g^j = -\partial_i\partial_k K^j/g^j = \partial_i\partial_j K^k/g^k = -\partial_j\partial_k K^i/g^i.$$

Hence $\partial_k\partial_j K^i = 0$ if i,j,k are different indices, so $\partial_k\partial_j F = 0$ when $j \neq k$, and

$$g^j\partial_j^2 F = g^j\partial_j^2\partial_i K^i = -\partial_j\partial_i^2 K^j g^i = -g^i\partial_i^2 F = g^k\partial_k^2 F = -g^j\partial_j^2 F.$$

Thus F is affine linear. When K is defined by (6.3.8) with $K_0 = 0$, then

$$\partial_j K^i g^j = 2\theta^j x_i - 2x_j\theta^i + 2A(x,\theta)\delta_{ij}g^j$$

since $A(x,\theta) = \sum x_j\theta^j/g^j$, so (6.3.7) holds with $F = 2A(x,\theta)$. (The coordinates of x should have been denoted by x^j for consistency but that would conflict with our notation elsewhere.) This is an arbitrary linear form. If $K_0 = \sum_0^n x_j\partial_j$ then (6.3.7) is valid with $F = 1$. In view of the linearity of (6.3.7) it just remains to study the case where $F = 0$, that is,

$$\partial_j K^i g^j + \partial_i g^i K^j = 0, \quad i,j = 0,\dots,n.$$

Thus $\partial_j(K^i/g^i)$ is a skew symmetric matrix Z_{ij}, and it is constant since

$$\partial_k\partial_j K^i/g^i = -\partial_k\partial_i K^j/g^j = -\partial_i\partial_k K^j/g^j,$$

which must vanish since the sign changes after three circular permutations of i, k, j. It follows that up to a constant vector field

$$\sum_{i=0}^{n} K^i\partial_i = \sum_{i,j=0}^{n} Z_{ij}x_j g^i\partial_i = \tfrac{1}{2}\sum_{i,j=0}^{n} Z_{ij}(x_j/g^j\,\partial_i - x_i/g^i\,\partial_j)g^i g^j$$

which is of the form (6.3.9).

Remark. That the vector fields in (6.3.8) appear is no surprise since they are obtained from constant vector fields by an inversion.

Now we choose $L = \sum_0^n \partial_i K^i/2 - F = (n-1)(A(\cdot,\theta) + c/2)$. Then the last term in (6.3.3)′ simplifies to

$$-(n-1)\sum_0^n \theta^k\partial_k u^2,$$

so (6.3.3)′ can be written

$$(6.3.3)'' \quad 2\Big(\sum_{i=0}^{n} K^i\partial_i u + (n-1)(A(x,\theta)+c/2)u\Big)\sum_{j,k=0}^{n} g^{jk}\partial_j\partial_k u$$
$$= \sum_{j=0}^{n}\partial_j\Big(\sum_{i=0}^{n} T_i^j(u)K^i + (n-1)\big((2A(x,\theta)u+cu)\sum_{k=0}^{n} g^{jk}\partial_k u - \theta^j u^2\big)\Big).$$

The vector field K_0 in (6.3.8) does not contribute much more than translations of the origin. In the following discussion we therefore take $K_0 = 0$. (Later on we shall let $K_0 = (1,0,\dots,0)$ to take advantage also of the energy estimate (6.3.2).) The vector field K is then in the span of the vector fields Z_{jk} and Z_0 in (6.2.8) and (6.2.9) (with (6.2.8) interpreted as (6.3.9) divided by 2 for general A). In fact, an easy calculation gives

$$(6.3.10) \qquad \langle K,\partial\rangle = A(x,\theta)Z_0 + \sum_{j,k=0}^{n}\theta^j x_k Z_{jk}.$$

Note that the coefficients are linear in x. It is also remarkable and important that K agrees with the vector field V in the proof of Lemma 6.3.1 if the vector K there is replaced by θ. Thus the present vector field K belongs to the closed light cone if θ is chosen in its interior; $K(x)$ is isotropic (or 0) if and only if x is isotropic (or 0). Taking for A the standard Lorentz form and $\nu = \theta = (1,0,\dots,0)$ we shall now compute the leading term

$$(6.3.11) \qquad e = \sum_{i,j=0}^{n} T_i^j K^i\nu_j = \sum_{i=0}^{n} T_i^0 K^i$$

which will appear when we integrate (6.3.3)″ with respect to $\vec{x}$ to derive an energy estimate. We know already that e is nonnegative. As already observed after the proof of Lemma 6.3.1 we have

$$K = (x_0^2 + \cdots + x_n^2, 2x_0x_1, \dots, 2x_0x_n).$$

If we write $x = (x_0, \vec{x})$, $gu' = U = (U_0, \vec{U})$, it follows that (see (6.3.5))

$$\begin{aligned} e = 2U_0A(U,K) - K_0A(U,U) &= 2U_0(U_0(x_0^2+|\vec{x}|^2) - 2x_0\langle \vec{U},\vec{x}\rangle) - (x_0^2+|\vec{x}|^2)(U_0^2-|\vec{U}|^2) \\ &= U_0^2(x_0^2+|\vec{x}|^2) - 4x_0\langle \vec{U},\vec{x}\rangle U_0 + x_0^2|\vec{U}|^2 + |\vec{x}|^2|\vec{U}|^2 \\ &= A(x,U)^2 + U_0^2|\vec{x}|^2 - 2x_0U_0\langle \vec{x},\vec{U}\rangle + x_0^2|\vec{U}|^2 + |\vec{x}|^2|\vec{U}|^2 - \langle \vec{x},\vec{U}\rangle^2 \\ &= A(x,U)^2 + \tfrac{1}{2}\sum_{j,k=0}^{n}(x_jU_k - x_kU_j)^2. \end{aligned}$$

If we return to $u' = AU$, we obtain

$$e = |\langle x, \partial u\rangle|^2 + |x \wedge A^{-1}\partial u|^2.$$

Since $Ax \wedge \partial u$ has the components $(\partial_k A\, \partial_j u - \partial_j A\, \partial_k u)/2$, we obtain

Lemma 6.3.4. *If K is defined by (6.3.8) with $K_0 = 0$ and $\theta = (1,0,\dots,0)$, and if g is the standard Lorentz form, then the energy form e defined by (6.3.11) can be written*

$$e = |Z_0u|^2 + \sum_{j<k}|Z_{jk}u|^2$$

where Z_{jk} and Z_0 are defined by (6.2.8) and (6.2.9). The norm of $\partial u \mapsto \langle K, \partial u\rangle$ with respect to the quadratic form e in ∂u is $\le |x|$.

Proof. The first statement was just proved, and it implies the second in view of (6.3.10).

Keeping the same assumptions on A and K we shall now prove the positivity of the complete energy expression which comes from (6.3.3)″:

Lemma 6.3.5. *If $n > 2$ and $u \in C_0^\infty$ we have*

$$1/41 \le \frac{\left(\int \left(e(u) + (n-1)(2x_0u\partial_0u - u^2)\right)d\vec{x}\right.}{\|Z_0u\|^2 + \sum_{j<k}\|Z_{jk}u\|^2 + \|(n-1)u\|^2} \le 2. \tag{6.3.12}$$

Proof. Writing $2x_0u\partial_0u = 2uZ_0u - \sum_1^n x_j\partial_j u^2$, we obtain

$$2\int x_0u\partial_0u\, d\vec{x} = 2\int uZ_0u\, d\vec{x} + n\|u\|^2.$$

Thus the numerator $E(u)$ in (6.3.12) can be written

$$E(u) = \|Z_0u + (n-1)u\|^2 + \sum_{j<k}\|Z_{jk}u\|^2, \tag{6.3.13}$$

which proves the upper bound in (6.3.12), also when $n = 2$. To obtain the lower bound it suffices to establish a bound for $\|u\|^2$. With polar coordinates $\vec{x} = r\omega$ we can write the integrand in E explicitly as follows by using the second expression for e given above

$$\begin{aligned} &(\partial_tu)^2(t^2+r^2) + 4tr\partial_tu\partial_ru + (t^2+r^2)|\partial_{\vec{x}}u|^2 + 2(n-1)tu\partial_tu - (n-1)u^2 \\ &\ge -(2r\partial_ru + (n-1)u)^2t^2/(t^2+r^2) + (t^2+r^2)(\partial u/\partial r)^2 - (n-1)u^2, \end{aligned}$$

with equality in the radial case for an appropriate choice of $\partial_t u$. (Note that we have $U = (\partial_t u, -\partial_{\tilde{x}} u)$.) Writing $r^{\frac{n-1}{2}} u = v$ to remove the factor r^{n-1} in the volume element, we obtain

$$E \geq \int d\omega \int \big(-4r^2t^2|\partial_r v|^2/(t^2+r^2) + (t^2+r^2)(\partial v/\partial r + (1-n)v/2r)^2 - (n-1)v^2\big)\, dr$$
$$= \int d\omega \int \big((r^2-t^2)^2(r^2+t^2)^{-1}(\partial v/\partial r)^2 + (n-1)(n-3)4^{-1}(1+t^2/r^2)v^2\big)\, dr,$$

after expansion and integration by parts. We know that this is still positive when $n = 2$ and $v = O(r)$ at 0, so the integral of the first term is at least equal to $\int (1+t^2/r^2)v^2\, dr/4$, which proves that $E \geq (n-2)^2\|u\|^2/4$ when $n > 2$. Since the inequalities

$$\|a+b\|^2 + c^2 \leq E, \quad \|b\|^2 \leq \kappa^2 E, \quad \kappa = 2(n-1)/(n-2) \leq 4$$

imply that $\|a\|^2 = \|a+b-b\|^2 \leq (1+\kappa)\|a+b\|^2 + (1+\kappa^{-1})\|b\|^2$, hence

$$\|a\|^2 + \|b\|^2 + c^2 \leq (1+\kappa)\|a+b\|^2 + (2+\kappa^{-1})\|b\|^2 + c^2 \leq (1+\kappa+2\kappa^2+\kappa)E \leq 41E,$$

the lower bound in (6.3.12) follows when $a = Z_0 u$, $b = (n-1)u$, $c^2 = \sum_{j<k}\|Z_{jk}u\|^2$.

Remark. It is easy to show that no positive lower bound exists in (6.3.12) when $n = 2$.

When u is a solution of $\Box u = 0$ with Cauchy data in C_0^∞ it follows from (6.3.3)″ that $E(u)$ is independent of x_0. In view of Lemma 6.3.5 we conclude that

$$\|Z_0 u(x_0,\cdot)\|^2 + \sum \|Z_{jk}u(x_0,\cdot)\|^2 + \|u(x_0,\cdot)\|^2$$

can be estimated for all x_0 by a constant times the value for $x_0 = 0$. More generally, if $\Box u = f$ we can use the identity (6.3.3)″ just as in the proof of (6.3.2) if we observe that by (6.3.10)

$$|\langle K, \partial u\rangle + (n-1)x_0 u|^2 = |x_0(Z_0 u + (n-1)u) + \sum_1^n x_k Z_{0k} u|^2$$
$$\leq |x|^2(|Z_0 u + (n-1)u|^2 + \sum_1^n |Z_{0k}u|^2).$$

In view of (6.3.13) it follows from (6.3.3)″ that

$$dE(x_0;u)/dx_0 \leq 2\|F(x_0,\cdot)\|E(x_0;u)^{\frac{1}{2}}$$

where $F(x) = |x|f(x)$. Hence

$$E(x_0;u)^{\frac{1}{2}} \leq E(0;u)^{\frac{1}{2}} + \int_0^{x_0} \|F(t,\cdot)\|dt. \tag{6.3.14}$$

Combined with (6.3.12) this gives improved control of the behavior of u at infinity when we have additional information on the decay of f.

(6.3.14) can be extended to operators with variable coefficients just as Proposition 6.3.2 extended (6.3.2). However, the hypotheses needed are far more complicated now so we shall postpone the discussion until we are ready for an application which justifies them. Instead we end this section with a brief sketch of how Proposition 6.3.2 leads to existence and uniqueness theorems for the Cauchy problem for linear second order hyperbolic equations. Local results follow from global ones, so we consider an equation

$$Lu = \sum_{j,k=0}^{n} g^{jk}(x)\partial_j\partial_k u(x) + \sum_{j=0}^{n} b^j(x)\partial_j u(x) + c(x)u(x) = f(x), \quad 0 \le x_0 \le T, \tag{6.3.15}$$

such that all derivatives of the coefficients are bounded in $[0,T]\times\mathbf{R}^n$ and $\sum |g^{jk}(x) - \lambda^{jk}| \le \frac{1}{2}$, where $\sum \lambda^{jk}\partial_j\partial_k = \Box$. If $M_1(x_0) = \|u(x_0,\cdot)\| + \|u'(x_0,\cdot)\|$, then it follows from (6.3.6) if $u \in C^\infty$ for $0 \le x_0 \le T$ and vanishes for large $\vec{x}$ that for such x_0

$$M_1(x_0) \le C(M_1(0) + \int_0^{x_0} (\|f(t,\cdot)\| + M_1(t))dt), \tag{6.3.16}$$

since

$$\|u(x_0,\cdot)\| \le \|u(0,\cdot)\| + \int_0^{x_0} \|\partial_t u(t,\cdot)\|dt.$$

Now we can apply *"Gronwall's lemma"*:

Lemma 6.3.6. *If φ, k and E are nonnegative, E is increasing and*

$$\varphi(t) \le E(t) + \int_0^t \varphi(\tau)k(\tau)\,d\tau, \quad 0 \le t \le T,$$

then

$$\varphi(t) \le E(t)\exp\Big(\int_0^t k(\tau)\,d\tau\Big), \quad 0 \le t \le T.$$

Proof. It is enough to prove this when $t = T$, and E may be replaced by $E(T)$ then, so we may assume that E is constant. Writing

$$F(t) = E + \int_0^t \varphi(\tau)k(\tau)\,d\tau$$

we have

$$F'(t) = \varphi(t)k(t) \le F(t)k(t),$$

since $\varphi \le F$, hence

$$F(t)\exp\Big(-\int_0^t k(\tau)\,d\tau\Big) \le F(0) = E,$$

which completes the proof.

(6.3.16) implies in view of Lemma 6.3.6 that with another constant C_0 we have

$$\sum_{|\alpha|\le 1} \|\partial^\alpha u(x_0,\cdot)\| \le C_0\Big(\sum_{|\alpha|\le 1} \|\partial^\alpha u(0,\cdot))\| + \int_0^{x_0} \|f(t,\cdot)\|dt\Big). \tag{6.3.17}$$

We claim that, more generally, for any integer $s \ge 0$ there is a constant C_s such that

$$\sum_{|\alpha|\le s+1} \|\partial^\alpha u(x_0,\cdot)\| \le C_s\Big(\sum_{|\alpha|\le s+1} \|\partial^\alpha u(0,\cdot)\| + \int_0^{x_0} \sum_{|\alpha|\le s} \|\partial^\alpha f(t,\cdot)\|dt\Big). \tag{6.3.18}$$

(We assume that $0 \le x_0 \le T$, and all constants may depend on T.) In the proof we may assume that $s > 0$ and note that

$$L\partial^\alpha u = \partial^\alpha f + [L, \partial^\alpha]u,$$

where the commutator $[L, \partial^\alpha]$ is of order $\le s+1$ when $|\alpha| \le s$. If we apply (6.3.17) to all such equations and write

$$M_s(f; x_0) = \sum_{|\alpha|\le s} \|\partial^\alpha f(x_0, \cdot)\|,$$

and similarly for u, it follows that

$$M_{s+1}(u; x_0) \le C_s'(M_{s+1}(u; 0) + \int_0^{x_0} (M_s(f; t) + M_{s+1}(u; t))\, dt).$$

By Gronwall's lemma this gives (6.3.18). Note that if x_0 has a fixed positive lower bound, then we can estimate $\sum_{|\alpha|<s} \|\partial^\alpha f(0, \cdot)\|$ by the integral in (6.3.18). Using the equation $Lu = f$ we can therefore restrict $\sum_{|\alpha|\le s+1} \|\partial^\alpha u(0, \cdot)\|$ to terms of order ≤ 1 with respect to x_0.

We may assume without restriction that the coefficient of ∂_0^2 in L is identically 1, for by our hypotheses both this coefficient and its reciprocal as well as their derivatives have uniform bounds. Then the commutator $[L, \partial^\alpha]$ is of order ≤ 1 with respect to x_0 if ∂^α has no such derivative. Hence we obtain by repeating the proof of (6.3.18)

$$(6.3.18)' \qquad \sum_{|\alpha|\le 1} \|\partial^\alpha u(x_0, \cdot)\|_{(s)} \le C_s(\sum_{|\alpha|\le 1} \|\partial^\alpha u(0, \cdot)\|_{(s)} + \int_0^{x_0} \|f(t, \cdot)\|_{(s)} dt),$$

where

$$\|v\|_{(s)} = ((2\pi)^{-n} \int |\hat{v}(\xi)|^2 (1 + |\xi|^2)^s\, d\xi)^{\frac{1}{2}}, v \in C_0^\infty(\mathbf{R}^n)$$

is equivalent to $\sum_{|\alpha|\le s} \|\partial_x^\alpha v\|$ when s is a positive integer (and $\|\cdot\|$ is the L^2 norm). As in Section 5.3 we denote by $H_{(s)}(\mathbf{R}^n)$ the Hilbert space of all temperate distributions v in $\mathbf{R}^n$ for which the Fourier transform $\hat{v}$ is a function such that $\|v\|_{(s)} < \infty$. The estimate (6.3.18)′ can be extended to all real s, but for the sake of simplicity we shall only do so when s is a negative integer. Then we define $U(x_0, \cdot) \in \mathcal{S}(\mathbf{R}^n)$ by

$$U(x_0, \cdot) = (1 - \Delta)^s u(x_0, \cdot), \quad \text{that is, } u(x_0, \cdot) = (1 - \Delta)^{-s} U(x_0, \cdot),$$

where Δ is the Laplace operator in $\mathbf{R}^n$. The estimate (6.3.18)′ remains valid for such functions, with s replaced by $-s$, which is a positive integer. Set

$$M(x_0) = \sum_{|\alpha|\le 1} \|\partial^\alpha U(x_0, \cdot)\|_{(-s)} = \sum_{|\alpha|\le 1} \|\partial^\alpha u(x_0, \cdot)\|_{(s)}.$$

Then

$$M(x_0) \le C_s(M(0) + \int_0^{x_0} \|LU(t, \cdot)\|_{(-s)} dt).$$

We have

$$f = Lu = L((1 - \Delta)^{-s} U) = (1 - \Delta)^{-s} LU + RU,$$

where R is a differential operator of order $1 - 2s$ and order ≤ 1 with respect to x_0. If we write $R = \sum \partial^\beta c_{\alpha\beta\gamma} \partial^\alpha \partial^\gamma$ with $|\beta| \le -s$, $|\gamma| \le -s$, $|\alpha| \le 1$, and $\beta_0 = \gamma_0 = 0$, we see that

$$\|RU(t, \cdot)\|_{(s)} \le \sum \|c_{\alpha\beta\gamma} \partial^\alpha \partial^\gamma U(t, \cdot)\| \le CM(t),$$

so we obtain

$$\|LU(t,\cdot)\|_{(-s)} \le \|f(t,\cdot)\|_{(s)} + CM(t),$$

and (6.3.18)′ follows as before.

The importance of the estimate (6.3.18)′ when $s < 0$ is primarily that it allows one to prove existence theorems in the spaces $H_{(s)}$ with $s > 0$. To show how that is done we first observe that if s is a positive integer we can apply (6.3.18)′ with s replaced by $-s-1$, L replaced by the adjoint L^* and x_0 replaced by $T - x_0$. This gives

$$\|\varphi(x_0,\cdot)\|_{(-s)} \le \sum_{|\alpha|\le 1} \|\partial^\alpha \varphi(x_0,\cdot)\|_{(-s-1)} \le C\int_{x_0}^T \|L^*\varphi(t,\cdot)\|_{(-s-1)}\,dt, \quad 0 \le x_0 \le T,$$

if $\varphi \in C_0^\infty(\mathbf{R}^{n+1})$ and $\varphi = 0$ for $x_0 > T$. If $f \in L^1([0,T]; H_{(s)}(\mathbf{R}^n))$, then

$$|(f,\varphi)| = \Big|\int_0^T (f(t,\cdot),\varphi(t,\cdot))\,dt\Big| \le C\int_0^T \|L^*\varphi(t,\cdot)\|_{(-s-1)}dt,$$

so by the Hahn-Banach theorem we can find $u \in L^\infty([-\infty,T]; H_{(s+1)}(\mathbf{R}^n))$ such that

$$(f,\varphi) = (u, L^*\varphi), \quad \text{if } \varphi(x_0,T) = 0 \text{ for } x_0 \ge T,$$

and $u = 0$ when $x_0 < 0$. Thus $Lu = f$ for $0 < x_0 \le T$, in the sense of distribution theory. Set $\partial_0 u = v$. If we single out the terms containing some factor ∂_0, then the differential equation gives

$$\partial_0 v + \sum_1^n a_j(x)\partial_j v + a_0 v \in L^\infty([-\infty,T]; H_{(s-1)}).$$

If we change variables so that the characteristics $dx_j/dx_0 = a_j(x)$ become straight lines, integrate the equation and return to the original variables, then we see that $\partial_0 u = v \in L^\infty([-\infty,T]; H_{(s-1)})$, hence using the equation we find that $\partial_0^k u \in L^\infty([-\infty,T]; H_{(s-k)}$ if $0 \le k \le s$ and f is say smooth and vanishes outside a compact set when $0 \le x_0 \le T$. For $s \ge 2$ we conclude that the Cauchy data of u when $x_0 = 0$ must vanish, and since (6.3.17) extends by continuity to all u with the derivatives of order ≤ 2 in L^2, the solution obtained does not depend on s, so it is in C^∞. Now it follows by continuity in view of (6.3.18) that for every f such that $\partial^\alpha f \in L^1([0,T]; L^2)$ for $|\alpha| \le s$ there is a solution of the equation $Lu = f$ with $\partial^\alpha u \in L^\infty([0,T]; L^2)$ when $|\alpha| \le s+1$ and Cauchy data 0. A solution with arbitrary Cauchy data is of course obtained if one chooses any function u_0 with the given Cauchy data and introduces $u - u_0$ as unknown instead of u.

The existence and uniqueness theorems we have now proved imply local existence and uniqueness theorems. In fact, if L is just given in a neighborhood Ω of the origin, with smooth coefficients, we can choose $\chi \in C_0^\infty(\Omega)$ with $0 \le \chi \le 1$ so that $\chi = 1$ in another neighborhood of the origin. Then $\chi(x)L(x,\partial) + (1-\chi(x))L(0,\partial)$ will satisfy the global hypotheses made above if the support of χ is small enough, and the global existence theorems proved above imply local existence theorems for L. If we have a solution of $Lu = 0$ in a neighborhood of 0 with vanishing Cauchy data when $x_0 = 0$, then we just introduce $x_0 + x_1^2 + \cdots + x_n^2$ as a new variable instead of x_0 to guarantee that $x_0 > 0$ in the support except at 0. If we then extend L to a global operator as just indicated we obtain a solution of the extended homogeneous equation when $0 \le x_0 \le T$ if T is small enough, and the Cauchy data vanish when $x_0 = 0$. This gives local uniqueness of solutions of the hyperbolic Cauchy problem. In what follows we shall take such results in the case of linear equations for granted; they can be proved in many different ways. (See e.g. Hörmander [4, Chapter XXIII].)

6.4. Interpolation and Sobolev inequalities. Already for the elementary existence theorems proved at the end of Section 4.2 we needed inequalities like (4.2.17)′ to estimate the norm of composite functions. In the case of several space variables we must use L^2 estimates, obtained from the energy integral method, rather than L^∞ estimates, so we need the following analogous and more general interpolation inequalities. They are special cases of those of Gagliardo [1] and Nirenberg [1].

Theorem 6.4.1. *Let $1 \le r \le \infty$, $1 \le q \le \infty$, and let m be an integer ≥ 2. If $u \in L^q(\mathbf{R}^n)$ and $\partial^\alpha u \in L^r(\mathbf{R}^n)$ when $|\alpha| = m$, then $\partial^\alpha u \in L^{p(\alpha)}(\mathbf{R}^n)$ for $|\alpha| \le m$, if*

$$m/p(\alpha) = (m - |\alpha|)/q + |\alpha|/r; \tag{6.4.1}$$

moreover, if $\|\cdot\|_s$ denotes the L^s norm, then

$$\sup_{|\alpha|=j} \|\partial^\alpha u\|_{p(\alpha)} \le 4^{|\alpha|(m-|\alpha|)}\big(\sup_{|\alpha|=m} \|\partial^\alpha u\|_r\big)^{|\alpha|/m}\|u\|_q^{(m-|\alpha|)/m}. \tag{6.4.2}$$

The general result will follow from the special case $m = 2$, $n = 1$. We begin with a simple lemma.

Lemma 6.4.2. *Let I be a finite interval on $\mathbf{R}$ of length $|I|$, and let $u \in L^q(I)$, $u'' \in L^r(I)$ for some $q, r \in [1, \infty]$. Then $u' \in L^p(I)$, $1 \le p \le \infty$, and*

$$\|u'\|_p|I|^{1-1/p} \le \|u''\|_r|I|^{2-1/r} + 4\|u\|_q|I|^{-1/q}. \tag{6.4.3}$$

Proof. Since the inequality is invariant under affine changes of variables, we may assume that $I = (-\frac{1}{2}, \frac{1}{2})$. It is then obvious that the strongest case of (6.4.3) is obtained when $r = q = 1$ and $p = \infty$. Assume that

$$\max_{x\in I} u'(x) = \int_I |u''|dx + M$$

for some $M > 0$. Then $u'(x) \ge M$ in I, and it follows that for some $x_0 \in \mathbf{R}$

$$|u(x)| \ge M|x - x_0|, \quad x \in I,$$

where $x_0 \in I$ if u has a zero in I, and $x_0 < -\frac{1}{2}$ ($x_0 > \frac{1}{2}$) if $u > 0$ ($u < 0$) in I. Now

$$2\int_I |x - x_0|\, dx = \int_I (|x - x_0| + |x + x_0|)\, dx \ge 2\int_I |x|\, dx = 1/2,$$

hence $\int_I |u|\, dx \ge M/4$, which proves the lemma. (The estimate (6.4.3) is easily seen to be optimal when $r = q = 1$ and $p = \infty$, but the estimate derived in Lemma 6.4.3 does not contain the best possible constants.)

Lemma 6.4.3. *Let $u \in L^q(\mathbf{R}_+)$, $u'' \in L^r(\mathbf{R}_+)$ where $q, r \in [1, \infty]$. If $2/p = 1/r + 1/q$, it follows that $u' \in L^p(\mathbf{R}_+)$ and that*

$$\|u'\|_p \le 4(\|u''\|_r\|u\|_q)^{\frac{1}{2}}. \tag{6.4.4}$$

Proof. In an interval $I \subset \mathbf{R}_+$ where the two terms in the right-hand side of (6.4.3) are equal, we can write (6.4.3) in the form

$$\|u'\|_{p,I} \le 4(\|u''\|_{r,I}\|u\|_{q,I})^{\frac{1}{2}}, \tag{6.4.4$'$}$$

where the norms are taken in I, for $2-1/r-1/q=2-2/p$ by hypothesis. If E is a disjoint union of intervals I_j for which (6.4.4)′ holds, then

$$\|u'\|_{p,E}^p \le 4^p \sum \|u''\|_{r,I_j}^{p/2}\|u\|_{q,I_j}^{p/2} \le 4^p \big(\sum \|u''\|_{r,I_j}^r\big)^{p/2r}\big(\sum \|u\|_{q,I_j}^q\big)^{p/2q}$$

by Hölder's inequality, for $p/2r+p/2q=1$. Hence (6.4.4)′ is valid with I replaced by E. (This is obvious when $p=q=r=\infty$.) If we consider intervals I with given left end point a, then the first term in (6.4.3) is the largest one if $|I|$ is large (unless $u'=0$ in (a,∞)), for otherwise

$$\|u'\|_{p,I} \le 8\|u\|_{q,I}|I|^{1/p-1/q-1} \to 0 \text{ as } |I|\to\infty,$$

since $1/p-1-1/q=(1/r-2-1/q)/2 \le -\frac{1}{2}$. Whenever the first term is the largest one then

$$\|u'\|_{p,I}|I|^{-1/p} \le 2\|u''\|_{r,I}|I|^{1-1/r},$$
$$4\|u\|_{q,I}|I|^{-1/q} \le \|u''\|_{r,I}|I|^{2-1/r},$$

which implies $u(a)=u'(a)=0$ if I can be arbitrarily small. If 0 is not a critical value of u this can never happen, so we get a sequence of intervals $I_j=[a_j,a_{j+1})$ for which (6.4.4)′ holds because the two terms in the right-hand side of (6.4.3) are equal. $a_j\to\infty$ since u and u' would vanish at an accumulation point by the argument just given. This proves (6.4.4) when 0 is not a critical value of u. Now the set of critical values of $e^x u(x)$ is of measure 0, and if ε is not a critical value then 0 is not a critical value for $u(x)-\varepsilon e^{-x}$. Hence (6.4.4) holds with $u(x)$ replaced by $u(x)-\varepsilon e^{-x}$ for a sequence of values of ε converging to 0, which proves (6.4.4) in general.

Proof of Theorem 6.4.1. Assuming at first that $n=1$ we set

$$M_j = \|u^{(j)}\|_{p(j)}.$$

If all M_j with $0<j<m$ are finite, then it follows from (6.4.4) that

$$M_j \le 4(M_{j+1}M_{j-1})^{\frac{1}{2}},$$

hence $A_j^2 \le A_{j+1}A_{j-1}$ if $A_j = 4^{j^2}M_j$. This logarithmic convexity implies that $A_j \le A_m^{j/m}A_0^{(m-j)/m}$ and gives (6.4.2). If we do not already know that $u^{(j)}\in L^{p(j)}$, we can at least by standard regularization reduce the proof to functions with $u^{(j)}\in L^\infty$ for all j. Choose $\chi\in C_0^\infty$ with $0\le\chi\le 1$ and $\chi=1$ in a neighborhood of 0 and set $u_\varepsilon(x)=\chi(\varepsilon x)u(x)$. Then

$$\|u_\varepsilon\|_q \le \|u\|_q, \quad \|u_\varepsilon^{(m)}\|_r \le \|u^{(m)}\|_r + O(\varepsilon^{1-1/r}) = O(1) \text{ as } \varepsilon\to 0.$$

Hence we conclude from the first part of the proof that $u^{(j)}\in L^{p(j)}$, and that (6.4.2) holds.

When the number n of variables is larger than 1, we conclude from Hölder's inequality as in the proof of Lemma 6.4.3 that (6.4.2) holds if all derivatives are taken with respect to the same variable x_k. By repeated use of that result we find that $\partial^\alpha u\in L^{p(\alpha)}$ for $|\alpha|\le m$. If we now introduce

$$M_j = \sup_{|\alpha|=j}\|\partial^\alpha u\|_{p(\alpha)}$$

and repeat the arguments used above in the one dimensional case, the estimate (6.4.2) follows in general.

Corollary 6.4.4. *If $u, v \in L^\infty(\mathbf{R}^n)$ and $\partial^\alpha u, \partial^\alpha v \in L^r(\mathbf{R}^n)$ when $|\alpha| = m$, then $\partial^\alpha(uv) \in L^r(\mathbf{R}^n)$ when $|\alpha| = m$, and*

$$\sum_{|\alpha|=m} \|\partial^\alpha(uv)\|_r \le C_m\Big(\sum_{|\alpha|=m} \|\partial^\alpha u\|_r \|v\|_\infty + \|u\|_\infty \sum_{|\alpha|=m} \|\partial^\alpha v\|_r\Big). \tag{6.4.5}$$

Proof. $\partial^\alpha(uv)$ consists of 2^m terms of the form $\partial^\beta u \partial^\gamma v$ with $|\beta| + |\gamma| = m$. With the notation in Theorem 6.4.1, $q = \infty$, we have $1/p(\beta) + 1/p(\gamma) = 1/r$, so the L^r norm of each term is at most $16^{|\beta||\gamma|} B^{|\beta|/m} C^{|\gamma|/m}$, where B and C are the two expressions on the right-hand side of (6.4.5). Hence we obtain (6.4.5) with $C_m = 2^m 2^{m^2}\binom{m+n-1}{m}$.

Remark. A more general version of the preceding estimates is sometimes useful: If $v_1, \dots, v_j \in L^\infty(\mathbf{R}^n)$ and $\partial^\alpha v_1, \dots, \partial^\alpha v_j \in L^r(\mathbf{R}^n)$, then

$$\|\partial^{\alpha_1} v_1 \cdots \partial^{\alpha_j} v_j\|_r \le 2^{jm^2/2} \max_{1\le i\le j} \prod_{k\ne i} \|v_k\|_\infty \sup_{|\alpha|=m} \|\partial^\alpha v_i\|_r, \quad \text{if } \sum_1^j |\alpha_i| = m. \tag{6.4.5$'$}$$

Again by Theorem 6.4.1 we have

$$\|\partial^{\alpha_i} v_i\|_{r/\lambda_i} \le 2^{m^2/2} A_i^{\lambda_i} B_i^{1-\lambda_i}, \quad A_i = \sup_{|\alpha|=m} \|\partial^\alpha v_i\|_r, \ B_i = \|v_i\|_\infty,$$

where $\lambda_i = |\alpha_i|/m$, thus $0 \le \lambda_i$ and $\sum \lambda_i = 1$. Hölder's inequality gives the bound $2^{jm^2/2} \prod A_i^{\lambda_i} B_i^{1-\lambda_i}$ for the left-hand side of (6.4.5)$'$. This convex function of λ in a simplex takes its maximum at a vertex, that is, when one $\lambda_i = 1$ and the others are 0, which proves (6.4.5)$'$.

Corollary 6.4.5. *Let $u \in L^\infty(\mathbf{R}^n, \mathbf{R}^N)$, let $F \in C^m(\mathbf{R}^N)$, and assume that $\partial^\alpha u \in L^r(\mathbf{R}^n)$ when $|\alpha| = m$. Then $\partial^\alpha F(u) \in L^r(\mathbf{R}^n)$ when $|\alpha| = m$, and*

$$\sup_{|\alpha|=m} \|\partial^\alpha F(u)\|_r \le C_m \sup_{1\le|\gamma|\le m} |F^{(\gamma)}(u)| \|u\|_\infty^{|\gamma|-1} \sup_{|\alpha|=m} \|\partial^\alpha u\|_r, \tag{6.4.6}$$

if $m > 0$, while for $m = 0$

$$\|F(u) - F(0)\|_r \le M\|u\|_r,$$

if M is a Lipschitz constant for F in the range of u.

Proof. It suffices to prove the estimate when u is smooth. We assume that $m > 0$, for the statement is obvious when $m = 0$. It is clear that $\partial^\alpha F(u)$ is a linear combination of terms of the form

$$F^{(\gamma)}(u)\partial^{\alpha_1} u_{i_1} \dots \partial^{\alpha_j} u_{i_j},$$

where $|\alpha_1| + \dots + |\alpha_j| = |\alpha|$, $|\alpha_k| > 0$ for $k = 1, \dots, j$, $|\gamma| = j$. By (6.4.5)$'$ the L^r norm of such a term can be estimated by

$$C^{|\gamma|} \sup |F^{(\gamma)}(u)| \|u\|_\infty^{|\gamma|-1} \sup_{|\alpha|=m} \|\partial^\alpha u\|_r,$$

which proves (6.4.6).

Remark. Note the somewhat surprising fact that the estimate is linear in the norms of the derivatives of u of order m once a bound for u is known. We shall actually need a more general version of (6.4.6) where $F \in C^m(\mathbf{R}^{n+m})$ also depends on x,

$$(6.4.6)' \quad \sup_{|\alpha|=m} \|\partial^\alpha(F(x,u(x)) - F(x,0))\|_r$$
$$\leq C'_m \sup_{\substack{1\leq|\gamma|\leq|\alpha| \\ |\alpha|+|\beta|=m}} \sup_{|v|\leq\|u\|_\infty} |\partial_x^\beta \partial_v^\gamma F(x,v)| \|u\|_\infty^{|\gamma|-1} \|\partial^\alpha u\|_r$$
$$+ C''_m \sup_{|\alpha|=m} \sup_{|v|\leq\|u\|_\infty} |\partial_x^\alpha \partial_v F(x,v)| \|u\|_r.$$

The terms in $\partial^\alpha(F(x,u(x)) - F(x,0))$ where no derivative falls on u are estimated by the last expression, and for the others the estimates in the proof of (6.4.6) can be applied with m replaced by $m - |\beta|$ if ∂^β acts directly on the first argument of F.

We shall also need *Sobolev's lemma.* The following version (see Aubin [1]) is very precise and of great geometrical interest.

Proposition 6.4.6. *If $u \in \mathcal{E}'(\mathbf{R}^n)$ and $\partial_j u \in L^1(\mathbf{R}^n)$, $j = 1, \dots, n$, then it follows that $u \in L^{n/(n-1)}(\mathbf{R}^n)$ and that*

$$(6.4.7) \qquad \Big(\int |u|^{n/(n-1)} dx\Big)^{(n-1)/n} \leq C_n \int \Big(\sum_1^n |\partial_j u|^2\Big)^{\frac{1}{2}} dx.$$

Here $C_n = (\omega_n/n)^{(n-1)/n}/\omega_n$ where ω_n is the area of the unit sphere S^{n-1}.

Proof. When $n = 1$ this is just the obvious statement that $|u| \leq \|u'\|_1/2$, so we assume that $n > 1$. By a regularisation we can reduce the proof to the case where $u \in C_0^\infty$. Then $|u|$ is Lipschitz continuous and $|u'| = ||u|'|$ almost everywhere. Hence it is no restriction to assume that $u \geq 0$. This will imply (6.4.7) even if $\partial_j u$ are only measures. In that generality the geometrical meaning is more clear: if u is the characteristic function of a set E with C^1 boundary, then u' is the unit normal to ∂E multiplied by the surface measure $d\sigma$, so (6.4.7) means that

$$\big(m(E)/(\omega_n/n)\big)^{(n-1)/n} \leq \sigma(\partial E)/\omega_n,$$

which is the isoperimetric inequality stating that among all sets with given volume the ball has the smallest boundary area. To prove (6.4.7) in general for $0 \leq u \in C_0^\infty$ we denote by χ_t the characteristic function of $\{x; u(x) > t\}$. Then $u = \int \chi_t dt$, so Minkowski's inequality gives

$$\|u\|_{n/(n-1)} \leq \int \|\chi_t\|_{n/(n-1)}\, dt.$$

When t is not a critical value of u, then χ_t is the characteristic function of a set with smooth boundary and, as just observed, the isoperimetric inequality gives

$$\|\chi_t\|_{n/(n-1)} \leq C_n \int |\chi'_t|\, dx,$$

where the right-hand side should be understood as the total mass of a measure. Now $\chi_t = H(u - t)$ where H is the Heaviside function, so $\chi'_t = u'\delta(u - t)$. Thus

$$\int |\chi'_t|\, dx = \int |u'|\delta(u - t)\, dx.$$

Since the set of critical values of u is closed and of measure 0, we obtain (6.4.7) now by integrating over the complement, for $\int \delta(u - t)\, dt = 1$.

To complete the proof we digress to give a beautiful proof of the Brunn-Minkowski inequality, following Federer [1]:

Proposition 6.4.7. *If A and B are compact subsets of $\mathbf{R}^n$ and $A+B$ denotes the sum $\{x+y; x\in A, y\in B\}$, then*

$$m(A+B)^{1/n} \ge m(A)^{1/n} + m(B)^{1/n}. \tag{6.4.8}$$

Proof. It suffices to prove this when A and B are unions of finitely many disjoint intervals, that is, sets of the form $\{x\in\mathbf{R}^n; \alpha_j \le x_j \le \beta_j \text{ for } j=1,\dots,n\}$. This can be done by induction over $a+b$ if a and b are the number of intervals which constitute A and B. In fact, if A and B are both intervals, with side lengths a_i, b_i, $i=1,\dots,n$, then

$$m(A)^{1/n}+m(B)^{1/n} = \prod_1^n a_j^{1/n} + \prod_1^n b_j^{1/n} \le (\sum_1^n a_j + \sum_1^n b_j)/n = 1 = \prod_1^n (a_j+b_j)^{1/n}$$

by the inequality between geometric and arithmetic means, if $a_j+b_j = 1$ for every j. By homogeneity reasons this gives (6.4.8) in general if A and B are both intervals. Now assume that $a>1$. Then there is a plane $x_j =$ constant separating two of the intervals defining A. In view of the translation invariance it is no restriction to assume that it is the plane $x_j=0$, and that

$$m(A_+)/m(A_-) = m(B_+)/m(B_-),$$

if $A_\pm$ and $B_\pm$ are the intersections of A and B with the half spaces defined by $x_j\ge 0$ and $x_j\le 0$ respectively. These are constituted by at most $a-1$ and at most b intervals, so by the inductive hypothesis

$$\begin{aligned} m(A+B) &\ge m(A_+ + B_+) + m(A_- + B_-)\\ &\ge (m(A_+)^{1/n} + m(B_+)^{1/n})^n + (m(A_-)^{1/n}+m(B_-)^{1/n})^n\\ &= (m(A)^{1/n}+m(B)^{1/n})^n \end{aligned}$$

which completes the proof.

If we take for A a set with C^2 boundary and for B the unit ball, then

$$m(A+rB) = m(A) + r\sigma(\partial A) + O(r^2),$$

hence

$$m(A+rB)^{1/n} - m(A)^{1/n} = r\sigma(\partial A)m(A)^{(1-n)/n}/n + O(r^2), \quad r>0,$$

and when $r\to 0$ it follows from (6.4.8) that $\sigma(\partial A)\ge nm(A)^{(n-1)/n}m(B)^{1/n}$, which is the isoperimetric inequality.

Returning to Sobolev's inequality we apply (6.4.7) to a power $|u|^{r+1}$ where $r\ge 0$. (We assume that u is smooth with compact support in the calculation.) This gives

$$\Big(\int |u|^{(r+1)n/(n-1)}dx\Big)^{(n-1)/n} \le C_n(1+r)\int |u'||u|^r\,dx.$$

Set $q=(r+1)n/(n-1)$ and apply Hölder's inequality with the exponents p and q/r, where

$$1/p + r/q = 1, \text{ that is, } 1/p - 1/q = 1/n.$$

Note that the condition $r\ge 0$ means that $p\ge 1$. Then we obtain

$$\|u\|_q^{r+1} \le C_n(1+r)\|u'\|_p\|u\|_q^r,$$

that is,

$$\|u\|_q \le C_n q(1-1/n)\|u'\|_p, \text{ if } 1/p = 1/q+1/n,\ 1\le p<q<\infty. \tag{6.4.9}$$

Iteration of this inequality gives the *Sobolev estimate*

$$\|u\|_q \le C_{n,m,q}\sum_{|\alpha|=m}\|\partial^\alpha u\|_p, \text{ if } 1/p = 1/q + m/n,\ 1\le p<q<\infty. \tag{6.4.10}$$

Hence $u\in\mathcal{E}'$ and $\partial^\alpha u\in L^p$ for $|\alpha|=m$ implies $u\in L^q$ then. When $p>n$ the estimate (6.4.9) is no longer applicable but there is a Hölder estimate instead:

Proposition 6.4.8. *If $u' \in L^p(\mathbf{R}^n)$ and $p > n$, then u is a continuous function and*

$$|u(x) - u(y)| \le C_{n,p}|x-y|^{1-\frac{n}{p}}\|u'\|_p. \tag{6.4.11}$$

Proof. By a regularization we reduce the proof to the case where $u \in C^\infty$. In view of the homogeneity under scale changes we may also assume that $|x-y| = 1$ in the proof. First we shall prove that

$$|u * \varphi(x) - u(x)\int \varphi(y)dy| \le C_{p,n}\|\varphi\|_\infty \|u'\|_p, \tag{6.4.12}$$

if $p > n$ and φ vanishes outside the unit ball. It is sufficient to do so when $x = 0$. With polar coordinates r, ω we have to estimate

$$I = \iint (u(r\omega) - u(0))\varphi(-r\omega)r^{n-1}\,dr\,d\omega.$$

Let $\partial\Phi(r,\omega)/\partial r = r^{n-1}\varphi(-r\omega)$ and $\Phi(r,\omega) = 0$ when $r > 1$. Then $|\Phi| \le \|\varphi\|_\infty$, and

$$I = -\iint \partial u(r\omega)/\partial r\,\Phi(r,\omega)\,dr\,d\omega.$$

Now

$$\|u'\|_p \ge \Big(\int_0^1 \int |\partial u/\partial r|^p r^{n-1}\,dr\,d\omega\Big)^{1/p},$$

so Hölder's inequality gives

$$|I| \le \|u'\|_p\|\varphi\|_\infty\Big(\int_0^1\int r^{(1-n)q/p}\,dr\,d\omega\Big)^{1/q}$$

where $1/q + 1/p = 1$, hence $1 + (1-n)q/p = q - nq/p = q(p-n)/p > 0$ if $p > n$. Thus the integral converges then and (6.4.12) is proved. Since

$$|(u*\varphi)'| = |u' * \varphi| \le \|\varphi\|_q\|u'\|_p, \quad \text{hence } |u*\varphi(x) - u*\varphi(y)| \le |x-y|\|\varphi\|_q\|u'\|_p,$$

it follows from (6.4.12) if φ is fixed with integral equal to 1 that for some other constant

$$|u(x) - u(y)| \le C_{p,n}\|u'\|_p, \quad \text{if } |x-y| = 1,$$

and this completes the proof.

Corollary 6.4.9. *If m is a positive integer and $n/m < p \le \infty$, then $\partial^\alpha u \in L^p(\mathbf{R}^n)$ for $|\alpha| \le m$ implies that u is a continuous function and that*

$$\sup|u| \le C_{n,p,m}\sum_{|\alpha|\le m}\|\partial^\alpha u\|_p. \tag{6.4.13}$$

Proof. It suffices to prove the estimate in the unit ball. Choose $\chi \in C_0^\infty(\mathbf{R}^n)$ equal to 1 in the unit ball and set $v = \chi u$. Then $v = u$ in the unit ball and $\partial^\alpha v \in L^p$ for $|\alpha| \le m$, with the norms estimated by the right-hand side of (6.4.13). If $p > n$ it follows from Proposition 6.4.8 that v is continuous and that (6.4.13) is valid for v, which proves the corollary for $m = 1$. If $m/n > 1/p$ but $p \le n$, we can choose $\tilde p < p$ so that $m/n > 1/\tilde p$. Then $\partial^\alpha v \in L^{\tilde p}$ for $|\alpha| \le m$, and if $1/q = 1/\tilde p - 1/n$ it follows from (6.4.9) that $\partial^\alpha v \in L^q$

for $|\alpha| \le m-1$, with the norms estimated by the right-hand side of (6.4.13), and we have $1/q < (m-1)/n$. Hence the corollary follows by induction with respect to m.

We shall often refer to (6.4.9), (6.4.10) and Corollary 6.4.9 as *Sobolev's lemma.* Note that (6.4.13) applied to $\chi(x)u(Rx)$ where $\chi \in C_0^\infty(\{x; |x|<1\})$ and $\chi(0)=1$ gives the frequently useful estimate

$$(6.4.13)' \quad R^{n/p}|u(0)| \le C'_{n,p,m} \sum_{|\alpha|\le m} R^{|\alpha|}\Big(\int_{|x|<R} |\partial^\alpha u(x)|^p\,dx\Big)^{1/p}, \quad \text{if } n/m < p \le \infty.$$

If Γ is an open cone we even have for the same exponents p

$$(6.4.13)'' \qquad R^{n/p}|u(0)| \le C'_{n,p,m,\Gamma} \sum_{|\alpha|\le m} R^{|\alpha|}\Big(\int_{|x|<R,x\in\Gamma} |\partial^\alpha u(x)|^p\,dx\Big)^{1/p}.$$

It suffices to prove this when $|x|<R$ in $\operatorname{supp} u$, for u can be replaced by $\chi(\cdot/R)u(\cdot)$. With polar coordinates $x=r\omega$, where $r>0$ and $|\omega|=1$, we have

$$u(0) = (-1)^m \int_0^R r^{m-1}\partial_r^m u(r\omega)dr/(m-1)!,$$

hence

$$u(0)\int_{\omega\in\Gamma} d\omega = (-1)^m \int_{\omega\in\Gamma}\int_0^R r^{m-1}\partial_r^m u(r\omega)\,dr\,d\omega/(m-1)!.$$

With $1/p+1/p'=1$ it follows from Hölder's inequality that

$$|u(0)| \le C\Big(\int_{\omega\in\Gamma}\int_0^R |\partial_r^m u(r\omega)|^p r^{n-1}\,dr\,d\omega\Big)^{1/p}\Big(\int_{\omega\in\Gamma}\int_0^R r^{(m-n/p)p'-1}\,dr\,d\omega\Big)^{1/p'},$$

which proves (6.4.13)″. (One can also deduce (6.4.13)″ from (6.4.13)′.) When $p=1$ we can take $m=n$: if Q is the parallelepiped $\{x\in\mathbf{R}^n; 0\le x_j\le a_j\}, j=1,\dots,n$, then

$$(6.4.13)''' \qquad \prod_1^n a_j \sup_Q |u| \le \sum_{\alpha_1,\dots,\alpha_n\in\{0,1\}} a^\alpha \int_Q |\partial^\alpha u(x)|\,dx.$$

This is an immediate consequence of the one dimensional case which follows by integrating the inequality $|u(x_1)| \ge \sup_{[0,a_1]}|u| - \int_0^{a_1}|u'(t)|\,dt$ from 0 to a_1.

Remark. From Proposition 6.4.8 one also obtains Hölder continuity of order $\gamma>0$ if $\gamma\le m-n/p$ and γ is not an integer. We leave the verification as an exercise.

We are now prepared for the proof of *local* existence and uniqueness theorems for solutions of nonlinear hyperbolic equations. Uniqueness for solutions with sufficiently high regularity is easily reduced to the linear case:

Theorem 6.4.10. *Let $u\in C^3$ be a solution of the differential equation*

$$(6.4.14) \qquad F(x,u,u',u'')=0$$

in a neighborhood of $0\in\mathbf{R}^{1+n}$, which is hyperbolic at the origin with respect to the plane $x_0=0$, that is, assume that

$$\sum_{j,k=0}^n F_{jk}(0,u(0),u'(0),u''(0))\xi_j\xi_k$$

is hyperbolic in the direction of the ξ_0 axis if $F_{jk} = \partial F/\partial u_{jk}$, $u'' = (u_{jk})$. Let $F \in C^2$. Then any other solution $v \in C^3$ of (6.4.14) with $\partial^\alpha u = \partial^\alpha v$ when $x_0 = 0$ and $|\alpha| \le 2$ is equal to u in a neighborhood of 0.

Proof. By Taylor's formula we can write

$$F(x,u,u',u'') - F(x,v,v',v'') = \sum_{|\alpha|\le 2} a_\alpha(x,u,u',\dots,v'')\partial^\alpha(u-v),$$

where $a_\alpha \in C^1$ as a function of $x,u,\dots,v''$. Hence $u-v$ satisfies a linear differential equation with C^1 coefficients which by hypothesis is hyperbolic with respect to the plane $x_0 = 0$ at 0. Since the Cauchy data are 0 it follows from the results on linear equations proved in Section 6.3 that $u - v = 0$ in a neighborhood of 0. (See the remarks at the end of Section 6.3.)

Remark. The regularity condition used here is not minimal. For example, if the equation is quasilinear, that is, linear in the highest derivatives, the same proof works if $u, v \in C^2$.

To prove local existence theorems we shall argue essentially as in Section 6.3 for the linear case, using the interpolation inequalities above. Again we state a result which is global in the space variables but implies an existence theorem which is local in all the variables. For the sake of simplicity we just consider a quasilinear equation at first. Extensions and consequences will be discussed after the proof.

Theorem 6.4.11. *Consider the Cauchy problem*

$$\Box u + \sum_{j,k=0}^{n} \gamma^{jk}(x,u,u')\partial_j\partial_k u = f(x,u,u'), \tag{6.4.15}$$

$$u(0,\cdot) = u_0, \quad \partial_0 u(0,\cdot) = u_1, \quad \text{if } x_0 = 0, \tag{6.4.16}$$

where γ^{jk} and f are C^∞ functions, $\gamma^{00} = 0$, $f(x,0,0) = 0$, $\sum|\gamma^{jk}| < 1/2$. We assume that f and all derivatives of f or of γ^{jk} are bounded. If $u_0 \in H_{(s+1)}(\mathbf{R}^n)$ and $u_1 \in H_{(s)}(\mathbf{R}^n)$ for some integer $s > (n+2)/2$, then the Cauchy problem has for some $T > 0$ a solution

$$u \in L^\infty([0,T]; H_{(s+1)}(\mathbf{R}^n)) \cap C^{0,1}([0,T]; H_{(s)}(\mathbf{R}^n)). \tag{6.4.17}$$

Here $C^{0,1}$ denotes the space of Lipschitz continuous functions so the second condition means that $\partial_0 u \in L^\infty([0,T]; H_{(s)})$. This implies that $u \in C^2([0,T]\times\mathbf{R}^n)$ and that $\partial^\alpha u$ is bounded when $|\alpha| \le 2$. The supremum of all such T is equal to the supremum of all T such that the Cauchy problem has a C^2 solution with $\partial^\alpha u$ bounded for $0 \le t \le T$ and $|\alpha| \le 2$.

Proof. We shall solve the Cauchy problem by successive approximation starting from the solution u^0 of $\Box u^0 = 0$ with Cauchy data (6.4.16). Since

$$\hat u^0(t,\xi) = \hat u_0(\xi)\cos(t|\xi|) + \hat u_1(\xi)\sin(t|\xi|)/|\xi|$$

and $(1+|\xi|^2)^{\frac12}|\sin(t|\xi|)| \le (|t\xi|^2 + |\xi|^2)^{\frac12} = |\xi|(1+t^2)^{\frac12}$, it is clear that (6.4.17) holds for u^0. The arguments will be simplified by assuming during the proof that $u_0, u_1 \in \mathcal{S}$, but only the norms corresponding to (6.4.17) will be used in our estimates.

By the results on linear equations in Section 6.3 we can define u^ν inductively by solving the Cauchy problem for

$$\Box u^\nu + \sum_{j,k=0}^{n} \gamma^{jk}(x, J_1 u^{\nu-1})\partial_j\partial_k u^\nu = f(x, J_1 u^{\nu-1}), \tag{6.4.18}$$

with the boundary condition (6.4.16). Here $J_1 u = (u, \partial u) = (u, \partial_0 u, \dots, \partial_n u)$ denotes the 1-*jet* of u in all the variables. Apart from that we shall only work with smoothness in the space variables. All derivatives of every u^ν are continuous functions of t with values in $H_{(s)}$ for arbitrary s. We shall estimate them for $0 \le t \le T$ assuming that we already know uniform bounds for the derivatives of order ≤ 2,

$$\sum_{|\alpha|\le 2} |\partial^\alpha u^\nu(t,\cdot)| \le M, \quad 0 \le t \le T. \tag{6.4.19}$$

We shall then prove that (6.4.19) follows inductively for small T and large M. After that we shall prove convergence of the sequence.

There is of course no difficulty in choosing M and T so that (6.4.19) holds when $\nu = 0$. To estimate

$$M_\nu(t) = \|u^\nu(t,\cdot)\|_{(s+1)} + \|\partial_t u^\nu(t,\cdot)\|_{(s)} = \|J_1 u^\nu(t,\cdot)\|_{(s)},$$

we shall apply the *a priori* estimate (6.3.6) to the equations obtained when ∂_x^α is applied to (6.4.18) for all α with $|\alpha| \le s$. This yields the equations

$$\begin{aligned}(\Box + \sum_{j,k} \gamma^{jk}(x, J_1 u^{\nu-1})\partial_j\partial_k)\partial_x^\alpha u^\nu = \partial_x^\alpha f(x, J_1 u^{\nu-1})\\ - \sum_{j,k} [\partial_x^\alpha, \gamma^{jk}(x, J_1 u^{\nu-1})]\partial_j\partial_k u^\nu.\end{aligned} \tag{6.4.20}$$

The terms in the sum on the right are linear combinations with bounded coefficients of terms of the form

$$(\partial_x^{\alpha'}\partial_x \gamma^{jk}(x, J_1 u^{\nu-1}))(\partial_x^{\alpha''}\partial_x \partial u^\nu), \quad |\alpha'| + |\alpha''| = |\alpha| - 1 \le s - 1.$$

Here we have used that $\gamma^{00} = 0$. For $0 \le t \le T$ it follows from (6.4.19) with ν replaced by $\nu - 1$ and the remark after Corollary 6.4.5 that

$$\|\partial_x \gamma^{jk}(x, J_1 u^{\nu-1})\|_\infty \le C(M); \; \|\partial_x^\alpha(\gamma^{jk}(x, J_1 u^{\nu-1}) - \gamma^{jk}(x,0))\|_2 \le C(M) M_{\nu-1}, \; |\alpha| \le s.$$

Corollary 6.4.4 applied to $\partial_x \gamma^{jk}(x, J_1 u^{\nu-1})$ and $\partial_x \partial u^\nu$, with $m = s-1$, now shows that the L^2 norm of the sum in the right-hand side of (6.4.20) can be estimated by $C(M)(M_\nu(t) + M_{\nu-1}(t))$. By the remark after Corollary 6.4.5 we have

$$\|\partial_x^\alpha f(x, J_1 u^{\nu-1})\|_2 \le C(M) M_{\nu-1}, \quad |\alpha| \le s,$$

so an application of the energy estimate (6.3.6) yields, again by using (6.4.19),

$$M_\nu(t) \le C e^{CMt}(M_\nu(0) + C(M)\int_0^t (M_\nu(\tau) + M_{\nu-1}(\tau))d\tau), \quad 0 \le t \le T. \tag{6.4.21}$$

Here $M_\nu(0) = M_0(0)$ is independent of ν. By Gronwall's lemma we conclude that

$$M_\nu(t) \le C e^{CMt}(M_0(0) + C(M)\int_0^t M_{\nu-1}(\tau)\,d\tau)\exp(tCC(M)e^{CMt}).$$

Let $A > CM_0(0)$ and let $A > M_0(t)$, $0 \le t \le T$. We can then choose $T_{M,A}$ with $0 < T_{M,A} \le T$ so that $M_0(t) \le A$, $0 \le t \le T_{M,A}$, and

$$M_\nu(t) \le A, \quad 0 \le t \le T_{M,A},$$

if this is true with ν replaced by $\nu - 1$. Now Sobolev's lemma (Corollary 6.4.9) shows that

$$\sum_{|\alpha|\le 2} |\partial^\alpha u^\nu(t,\cdot)| \le C(A) \quad \text{if } M_\nu(t) \le A,$$

for $s > n/2 + 1$ so we can estimate the maximum of $J_1 u^\nu$ and $\partial_x(J_1 u^\nu)$, and $\partial_0^2 u$ is then estimated using (6.4.15). With $M = C(A)$ and T replaced by $T_{M,A}$ the estimate (6.4.19) is therefore also proved inductively, so we have found M and $T > 0$ such that for all ν we have (6.4.19) and

$$M_\nu(t) \le A, \quad 0 \le t \le T. \tag{6.4.22}$$

We shall now prove that u^ν converges in

$$C([0,T]; H_{(1)}(\mathbf{R}^n)) \cap C^1([0,T]; H_{(0)}(\mathbf{R}^n))$$

to a limit u which must then automatically satisfy (6.4.15), (6.4.16), (6.4.17). To do so we subtract two successive equations (6.4.18) and obtain

$$(\Box + \sum_{j,k} \gamma^{jk}(x, J_1 u^\nu)\partial_j\partial_k)(u^{\nu+1} - u^\nu) = \sum_{j,k} (\gamma^{jk}(x, J_1 u^{\nu-1}) \\ - \gamma^{jk}(x, J_1 u^\nu))\partial_j\partial_k u^\nu + f(x, J_1 u^\nu) - f(x, J_1 u^{\nu-1}).$$

Recall that we have uniform bounds for the derivatives of u^ν of order ≤ 2 for all ν. The L^2 norm of the right-hand side can therefore be estimated by a constant times

$$m_\nu(t) = \|u^\nu(t,\cdot) - u^{\nu-1}(t,\cdot)\|_{(1)} + \|\partial_t(u^\nu(t,\cdot) - u^{\nu-1}(t,\cdot))\|_{(0)} = \|J_1(u^\nu - u^{\nu-1})(t,\cdot)\|.$$

We have uniform bounds for the derivatives of γ^{jk}, so (6.3.17) yields

$$m_{\nu+1}(t) \le C\int_0^t m_\nu(\tau)\,d\tau, \quad 0 \le t \le T, \tag{6.4.23}$$

since the Cauchy data of $u^{\nu+1} - u^\nu$ vanish. Hence we obtain inductively

$$m_\nu(t) \le (Ct)^\nu \sup m_0/\nu!.$$

which completes the existence proof when the Cauchy data are in $\mathcal{S}$. This condition is eliminated by a standard approximation argument.

Let T_s be the supremum of all T such that the Cauchy problem has a solution satisfying (6.4.17). By Sobolev's lemma (6.4.17) implies uniform Hölder continuity with respect to x of $J_1 u$ and $\partial_x J_1 u$ when $0 \le t \le T$, and the differential equation then shows that $\partial_t^2 u(t,\cdot)$ is also uniformly Hölder continuous. This implies that $\partial^\alpha u$ is uniformly continuous and uniformly bounded for $0 \le t \le T$ if $0 \le |\alpha| \le 2$. What remains is therefore to show that there is no uniform bound for $0 \le t < T_s$. Assume the contrary, so that we have (6.4.19) for a fixed M and every $T < T_s$. Then the inequality (6.4.21) holds for $0 \le t < T_s$ with $M_\nu(t) = M(t)$ independent of ν, for we can take the sequence u^ν constantly equal to the solution of the exact Cauchy problem. But then it follows from Gronwall's lemma that $M(t)$ is uniformly bounded for $0 \le t < T_s$. As we have just seen it follows that u has a C^2 extension to $0 \le t \le T_s$, and we have $u(T_s,\cdot) \in H_{(s+1)}$, $\partial_t u(T_s,\cdot) \in H_{(s)}$. Hence the Cauchy problem has a solution with these data when $t = T_s$, satisfying (6.4.17) up to some time $T > T_s$, which contradicts the definition of T_s and completes the proof.

Remark 1. The proof of Theorem 6.4.11 remains valid with an obvious modification if we replace the hypothesis $f(x, 0, 0) = 0$ by the assumption that the support is compact. This observation is useful because it allows one to deduce a general local existence theorem for an equation

$$\sum_{j,k=0}^{n} g^{jk}(x, u, u')\partial_j\partial_k u = f(x, u, u')$$

which is hyperbolic with respect to the plane $x_0 = 0$ at 0, for the given Cauchy data. After a linear change of variables preserving the plane $x_0 = 0$ the equation is then of the form (6.4.15) with $\gamma^{jk} = 0$ at 0. Cutting γ^{jk} and f off in a suitable way in all variables x, u, u' we obtain an equation satisfying the hypotheses of Theorem 6.4.10 as weakened at the beginning of this remark.

Remark 2. Using the equation (6.4.15) we can get bounds in L^2 for all derivatives of u of order $\le s+1$ and not only for those which are of order ≤ 1 with respect to t. In fact, if $|\alpha| + k \le s$ we can write $\partial_x^\alpha \partial_t^k \partial u$ as a linear combination of terms of the form

$$a(x, J_1 u)\partial_x^{\alpha_1} J_1 u \ldots \partial_x^{\alpha_j} J_1 u, \quad |\alpha_1| + \cdots + |\alpha_j| \le |\alpha| + k.$$

This follows if we keep replacing $\partial_t^2 u$ by the expression given by the equation (6.4.15) until all terms are of order at most equal to 1 with respect to t. Using $(6.4.5)'$ we can then improve (6.4.17) to

$$\partial^\alpha u \in L^\infty([0, T]; L^2), \quad |\alpha| \le s + 1.$$

Remark 3. The proof of Theorem 6.4.11 shows that given T we have a solution of the Cauchy problem for $0 \le t \le T$ satisfying (6.4.17) provided that $\|u_0\|_{(s+1)}$ and $\|u_1\|_{(s)}$ are small enough, for some $s > (n+2)/2$. The relations between the size of the Cauchy data and the lifespan will be the subject of Section 6.5. Here we observe that in view of the finite propagation speed for solutions of hyperbolic equations it follows easily that the unboundedness of second order derivatives proved in Theorem 6.4.11 when t approaches the lifespan T_s for a solution satisfying (6.4.17) does not occur at infinity. Thus the solution does not have a C^2 extension beyond the time T_s at some point. From the discussion of first order systems in Chapter II and the reduction of second order equations in one space variable to first order systems given in the introduction, we see that in general first order derivatives may remain bounded up to the time T_s.

Remark 4. Theorem 6.4.11 and the preceding remarks remain valid if u takes its values in a finite dimensional vector space $\mathbf{R}^N$, the coefficients γ^{jk} are diagonal $N \times N$ matrices and f takes its values in $\mathbf{R}^N$. In particular, we can apply this remark to find a solution of a fully non-linear equation

$$(6.4.15)' \qquad \Box u = F(x, u, u', u'')$$

with Cauchy data (6.4.16). For the sake of simplicity we assume again that u is just a real valued function, and by solving for $\partial_0^2 u$ we can attain that ∂_0^2 only occurs in $\Box u$. Differentiation of the equation $(6.4.15)'$ with respect to x_j gives a quasilinear equation for $u_j = \partial_j u$, $j = 0, \ldots, n$, which together with $(6.4.15)'$ can be considered as a quasilinear system of equations for $(u, \partial_0 u, \ldots, \partial_n u)$. (The second order derivatives occurring as arguments of F in $(6.4.15)'$, for example, are just first order derivatives of our new unknowns u_j.) From the given Cauchy data for u and the differential equation we can calculate Cauchy data for all u_j. If we assume one derivative more for the initial data than in the quasilinear case we therefore get a solution to our new system, and a vector valued version of Theorem 6.4.10 shows that $u_j = \partial_j u$ for the solution obtained. Thus Theorem 6.4.11 remains valid in the fully non-linear case if we require $s > (n+4)/2$; the supremum of all T such that a solution satisfying (6.4.17) exists in $[0,T]$ is equal to the supremum of all T such that there is a C^3 solution with $\partial^\alpha u$ bounded for $0 \le t \le T$ and $|\alpha| \le 3$.

If one is content with existence in a time interval depending on bounds for a larger number of derivatives of u_0 and u_1, then a much more elementary argument can be used, as pointed out by Klainerman [3]. We shall use his idea in the proof of global existence theorems to avoid the more intricate interpolation inequalities which would otherwise be needed then since one cannot confine oneself to discussing regularity in the space variables for fixed time. As a preparation we shall now present the idea as an alternative proof of Theorem 6.4.11.

With a positive integer s to be chosen later we shall try to establish uniform bounds for

(6.4.24) $$M_\nu(t) = \sum_{|\alpha| \le s+1} \|\partial^\alpha u^\nu(t,\cdot)\|,$$

when $0 \le t \le T$. Note that we now consider *all* derivatives of order $\le s+1$. By Sobolev's inequality, applied for fixed t, it follows that

(6.4.25) $$|\partial^\alpha u^\nu(t,\cdot)| \le C M_\nu(t), \quad |\alpha| + \kappa \le s+1,$$

if κ is the smallest integer $> n/2$. This estimate is applicable to all α with $|\alpha| \le 2$ if $s \ge \kappa + 1$. We want to prove inductively that

(6.4.26) $$M_\nu(t) \le M, \quad 0 \le t \le T.$$

Assume that this is already known with ν replaced by $\nu - 1$, and consider (6.4.20) again, now with ∂_x^α replaced by any ∂^α with $|\alpha| \le s$. In the last sum there is no term where $J_1 u^\nu$ or $J_1 u^{\nu-1}$ is differentiated more than s times. If N is an integer with $2(N+1) > s+1$, that is, $2N \ge s$, then no term contains two factors where Ju^ν or $Ju^{\nu-1}$ is differentiated more than N times. Thus (6.4.25) can be used in all factors except one if $N + \kappa \le s$, so we conclude that the L^2 norm of the right-hand side of (6.4.20) can be estimated by $C(M)(M_\nu(t)+1)$, and the energy estimates give

$$M_\nu(t) \le Ce^{CMt}(M_\nu(0) + C(M)\int_0^t (M_\nu(\tau)+1)d\tau).$$

When $\nu \ge s$ the derivatives of $\partial^\alpha u^\nu$, $|\alpha| \le s$, are equal to the derivatives of the formal solution of the Cauchy problem when $t = 0$, hence independent of ν, and we conclude using Gronwall's lemma that

$$M_\nu(t) \le Ce^{CMt}(M_\nu(0) + C(M)t)\exp(tCC(M)e^{CMt}), \quad 0 \le t \le T.$$

If $M > CM_\nu(0)$ and T is small enough it follows that (6.4.26) is valid, and the proof is finished as before.

The conditions $N + \kappa \le s \le 2N$ imply $N \ge \kappa$ and $s \ge 2\kappa$, so to estimate u^ν we must in this approach assume bounds on about twice as many derivatives of the initial data as in Theorem 6.4.11 . However, we shall use it all the same in the following section because of the simplicity of the argument.

6.5. Global existence theorems for nonlinear wave equations. In this section we shall study the Cauchy problem in $\mathbf{R}^{1+n}$

(6.5.1) $$\sum_{j,k=0}^n g^{jk}(u')\partial_j\partial_k u = f(u'),$$

(6.5.2) $$u(0,x) = \varepsilon u_0(x), \quad \partial_0 u(0,x) = \varepsilon u_1(x).$$

or rather use $t = x_0$, $q = r - x_0$, and ω as new variables. Note that this makes the light cone a coordinate plane $q = 0$. The vector fields (6.2.8) with $j, k \neq 0$ annihilate t and q and preserve homogeneity, so they can be regarded as vector fields in S^{n-1}. Since the orthogonal covectors are spanned by dr and dt, they span the vector fields in the unit sphere. Let us denote this set of vector fields by Ω. The remaining vector fields (6.2.8), (6.2.9) are

$$Z_{0k} = x_0\partial_k + x_k\partial_0, \quad 0 < k \le n; \quad \text{and } Z_0 = \sum_0^n x_j\partial_j.$$

Since $\partial/\partial r$ becomes $\partial/\partial q$ and $\partial/\partial x_0$ becomes $\partial/\partial t - \partial/\partial q$ in the new coordinates, we obtain

$$\sum_1^n \omega_k Z_{0k} = t\partial/\partial q + (t+q)(\partial/\partial t - \partial/\partial q) = (t+q)\partial/\partial t - q\partial/\partial q.$$

The radial vector field Z_0 is $t\partial/\partial t + q\partial/\partial q$ with these coordinates. It follows that in the conic neighborhood of Λ where $t/2 < |x| < 3t/2$ we have

$$t\partial/\partial t = a_0 Z_0 + \sum_1^n a_k Z_{0k}, \quad q\partial/\partial q = b_0 Z_0 + \sum_1^n b_k Z_{0k}, \quad \partial/\partial q = \sum_1^n \omega_j\partial_j,$$

where a_ν, b_ν and of course ω_j are homogeneous of degree 0. Writing $u(t,x) = v(t,q,\omega)$ we obtain

$$t^{n-1} \sum_{\alpha+\beta+|\gamma|\le N} \int_{|q|<t/2} |(q\partial/\partial q)^\alpha(\partial/\partial q)^\beta(\partial/\partial\omega)^\gamma v(t,q,\omega)|^2\, dq\, d\omega \tag{6.5.4}$$
$$\le C \sum_{|I|\le N} \|Z^I u(t,\cdot)\|^2,$$

where $N = (n+2)/2$, for the Lebesgue measure becomes $r^{n-1}drd\omega = (t+q)^{n-1}dqd\omega$. Taking $\alpha = 0$ we conclude using Sobolev's lemma that $t^{n-1}|v(t,q,\omega)|^2$ can be estimated by the right-hand side if $|q| < t/4$. (We can use local coordinates on the unit sphere, and $t > 2/5$ since $|x| + t \ge 1$ and $|x| < 3t/2$.)

Let $\chi \in C_0^\infty((-1/2,1/2))$, $\chi(0) = 1$, and set

$$V_Q(t,q,\omega) = \chi((q-Q)/Q)v(t,q,\omega)$$

for some Q with $1 < |Q| < t/4$. We have $V_Q(t,Q,\omega) = v(t,Q,\omega)$, and $|q - Q| < |Q|/2$, hence $|Q|/2 < |q| < 3|Q|/2 < t/2$ in the support. Hence the square of the L^2 norm of

$$(Q\partial/\partial q)^\alpha(\partial/\partial\omega)^\gamma V_Q(t,q,\omega) = (\frac{Q}{q}q\partial/\partial q)^\alpha(\partial/\partial\omega)^\gamma\chi((q-Q)/Q)v(t,q,\omega),$$

can be estimated by a constant times the sum in the left-hand side of (6.5.4) with $\beta = 0$ when $\alpha + |\gamma| \le N$, for

$$(q\partial/\partial q)((q-Q)/Q) = q/Q, \quad (q\partial/\partial q)(Q/q) = -Q/q,$$

are uniformly bounded in the support. Taking q/Q as a new variable instead of q we obtain another factor Q from the integration element, and Sobolev's lemma gives

$$t^{n-1}Q|v(t,Q,\omega)|^2 \le C \sum_{|I|\le(n+2)/2} \|Z^I u(t,\cdot)\|^2.$$

or rather use $t = x_0$, $q = r - x_0$, and ω as new variables. Note that this makes the light cone a coordinate plane $q = 0$. The vector fields (6.2.8) with $j, k \neq 0$ annihilate t and q and preserve homogeneity, so they can be regarded as vector fields in S^{n-1}. Since the orthogonal covectors are spanned by dr and dt, they span the vector fields in the unit sphere. Let us denote this set of vector fields by Ω. The remaining vector fields (6.2.8), (6.2.9) are

$$Z_{0k} = x_0\partial_k + x_k\partial_0, \quad 0 < k \leq n; \quad \text{and } Z_0 = \sum_0^n x_j\partial_j.$$

Since $\partial/\partial r$ becomes $\partial/\partial q$ and $\partial/\partial x_0$ becomes $\partial/\partial t - \partial/\partial q$ in the new coordinates, we obtain

$$\sum_1^n \omega_k Z_{0k} = t\partial/\partial q + (t+q)(\partial/\partial t - \partial/\partial q) = (t+q)\partial/\partial t - q\partial/\partial q.$$

The radial vector field Z_0 is $t\partial/\partial t + q\partial/\partial q$ with these coordinates. It follows that in the conic neighborhood of Λ where $t/2 < |x| < 3t/2$ we have

$$t\partial/\partial t = a_0 Z_0 + \sum_1^n a_k Z_{0k}, \quad q\partial/\partial q = b_0 Z_0 + \sum_1^n b_k Z_{0k}, \quad \partial/\partial q = \sum_1^n \omega_j \partial_j,$$

where a_ν, b_ν and of course ω_j are homogeneous of degree 0. Writing $u(t,x) = v(t,q,\omega)$ we obtain

$$(6.5.4) \qquad t^{n-1} \sum_{\alpha+\beta+|\gamma|\leq N} \int_{|q|<t/2} |(q\partial/\partial q)^\alpha (\partial/\partial q)^\beta (\partial/\partial\omega)^\gamma v(t,q,\omega)|^2 \, dq\, d\omega \leq C \sum_{|I|\leq N} \|Z^I u(t,\cdot)\|^2,$$

where $N = (n+2)/2$, for the Lebesgue measure becomes $r^{n-1} dr d\omega = (t+q)^{n-1} dq d\omega$. Taking $\alpha = 0$ we conclude using Sobolev's lemma that $t^{n-1}|v(t,q,\omega)|^2$ can be estimated by the right-hand side if $|q| < t/4$. (We can use local coordinates on the unit sphere, and $t > 2/5$ since $|x| + t \geq 1$ and $|x| < 3t/2$.)

Let $\chi \in C_0^\infty((-1/2, 1/2))$, $\chi(0) = 1$, and set

$$V_Q(t,q,\omega) = \chi((q-Q)/Q) v(t,q,\omega)$$

for some Q with $1 < |Q| < t/4$. We have $V_Q(t,Q,\omega) = v(t,Q,\omega)$, and $|q - Q| < |Q|/2$, hence $|Q|/2 < |q| < 3|Q|/2 < t/2$ in the support. Hence the square of the L^2 norm of

$$(Q\partial/\partial q)^\alpha (\partial/\partial\omega)^\gamma V_Q(t,q,\omega) = \left(\frac{Q}{q} q\partial/\partial q\right)^\alpha (\partial/\partial\omega)^\gamma \chi((q-Q)/Q) v(t,q,\omega),$$

can be estimated by a constant times the sum in the left-hand side of (6.5.4) with $\beta = 0$ when $\alpha + |\gamma| \leq N$, for

$$(q\partial/\partial q)((q-Q)/Q) = q/Q, \quad (q\partial/\partial q)(Q/q) = -Q/q,$$

are uniformly bounded in the support. Taking q/Q as a new variable instead of q we obtain another factor Q from the integration element, and Sobolev's lemma gives

$$t^{n-1} Q |v(t,Q,\omega)|^2 \leq C \sum_{|I|\leq(n+2)/2} \|Z^I u(t,\cdot)\|^2.$$

This completes the proof.

Remark. If u is a solution of the homogeneous unperturbed wave equation with Cauchy data in C_0^∞, then we know from (6.3.2) that $\|Z^I u'(t,\cdot)\|$ is uniformly bounded for any I, for $\Box Z^I u = 0$, too. (See Section 6.2.) By Proposition 6.5.1 this implies

$$\sup(1+|t|)^{\frac{n-1}{2}}(1+||x|-|t||)^{\frac{1}{2}}|u'(t,x)| < \infty.$$

The weak Huygen's principle gives $u(t,x) = 0$ if $|x|-|t| >$ constant, so it follows that

$$\sup(1+|t|)^{\frac{n-1}{2}}(1+||x|-|t||)^{-\frac{1}{2}}|u(t,x)| < \infty.$$

From Section 6.2 we know that these estimates have the right order of magnitude near the boundary of the light cone, that is, when $|x|-|t|$ is bounded. However, the best bounds are

$$\sup(1+|t|)^{\frac{n-1}{2}}(1+||x|-|t||)^{\frac{n-1}{2}}(|u(t,x)|+|u'(t,x)|) < \infty,$$

and they cannot be obtained from L^2 estimates of $Z^I u$. Fortunately this flaw will not affect the following existence theorems much since the critical estimates concern the immediate neighborhood of the boundary of the light cone.

Theorem 6.5.2. *The Cauchy problem* (6.5.1), (6.5.2) *with* $u_j \in C_0^\infty(\mathbf{R}^n)$ *has a* C^∞ *solution for* $t \ge 0$ *if* $n \ge 4$ *and* ε *is sufficiently small.*

Proof. We know from Theorem 6.4.11 that the set of all T such that a smooth solution exists for $0 \le t \le T$ is open. The theorem will be proved by establishing estimates of the solution which are independent of T and imply that this set is also closed.

As at the end of Section 6.4 we choose positive integers s and N with

$$N + \kappa \le s \le 2N, \tag{6.5.5}$$

where κ is the smallest integer $> n/2$. We want to prove that there is a constant M such that for small ε

$$M_s(t) = \sum_{|I|\le s} \|Z^I u'(t,\cdot)\| \le M\varepsilon, \quad \text{if } 0 \le t \le T. \tag{6.5.6}$$

This is true for small T if M is large enough. We shall prove that for sufficiently large M the estimate (6.5.6) implies the same bound with M replaced by $M/2$. By "continuous induction" we may then conclude that (6.5.6) holds.

By Proposition 6.5.1 it follows from (6.5.6) that

$$(1+t)^{\frac{n-1}{2}}|Z^I u'(t,x)| \le CM\varepsilon, \quad \text{if } 0 \le t \le T,\ |I| + \kappa \le s. \tag{6.5.7}$$

Since $N + \kappa \le s$ we can use this estimate when $|I| \le N$. To estimate $\|Z^I u'(t,\cdot)\|$ when $|I| \le s$ we apply Z^I to (6.5.1) and obtain (cf. (6.4.20)), with the notation $\gamma^{jk}(u') = g^{jk}(u') - g^{jk}(0)$,

$$\begin{aligned}(\Box + \sum_{j,k} \gamma^{jk}(u')\partial_j\partial_k)Z^I u &= Z^I f(u') + [\Box, Z^I]u \\ &- \sum_{j,k}[Z^I, \gamma^{jk}(u')]\partial_j\partial_k u - \sum_{j,k}\gamma^{jk}(u')[Z^I, \partial_j\partial_k]u.\end{aligned} \tag{6.5.8}$$

From (6.5.7) we obtain in particular

$$|\gamma^{jk}(u')| \le CM\varepsilon(1+t)^{\frac{1-n}{2}} \le CM\varepsilon,$$

so the hypotheses of the energy estimate (6.3.6) are fulfilled for small ε. Moreover, if $\gamma^{jk}(u')$ is considered as a function of t, x we have with the notation used there

$$|\gamma'(t)| \le CM\varepsilon(1+|t|)^{\frac{1-n}{2}}. \tag{6.5.9}$$

This is an integrable function when $n > 3$, which motivates this condition in the theorem. The L^2 norm of the last sum in (6.5.8) can be estimated by $CM\varepsilon(1+t)^{\frac{1-n}{2}}M_s(t)$, for $[Z^I, \partial_j\partial_k]$ is a linear combination of operators $Z^J\partial_i$ with $|J| \le s$ since $[Z, \partial_i]$ is always either 0 or equal to $\pm\partial_k$ for some k. We can write

$$[Z^I, \gamma^{jk}(u')]\partial_j\partial_k u$$

as a sum of derivatives of γ^{jk} multiplied by components of

$$Z^{J_1}u' \cdots Z^{J_r}u' Z^K \partial_j\partial_k u$$

where $|J_1| + \cdots + |J_r| + |K| \le s$, $J_i \ne 0$ for every i, and $r \ne 0$. At most one of the positive integers $|J_1|, \ldots, |J_r|, |K| + 1$ can be larger than N since $2(N+1) > s+1$ by (6.5.5), so we can estimate all factors except one using (6.5.7). There are at least two factors. For small ε it follows that the L^2 norm of the first sum of commutators in (6.5.8) can also be estimated by $CM\varepsilon(1+t)^{\frac{1-n}{2}}M_s(t)$. Since f vanishes of second order at 0, we can by Taylor's formula write

$$f(u') = \sum_{j,k} f_{jk}(u')\partial_j u \partial_k u$$

with smooth f_{jk}, and a similar estimate is then obtained for this term. Finally

$$[\Box, Z^I] = \sum_{|J|<|I|} c_{I,J} Z^J \Box u$$

with constant $c_{I,J}$, for $[\Box, Z]$ is equal to 0 or $2\Box$ for each factor. We express $Z^J\Box u$ by means of the equation (6.5.1),

$$Z^J\Box u = Z^J f(u') - Z^J \sum_{j,k} \gamma^{jk}(u')\partial_j\partial_k u.$$

Since $|J| < s$ all terms obtained are of the form already discussed. Hence it follows from (6.3.6) when ε is small that

$$\|\partial Z^I u(t,\cdot)\| \le 3\|\partial Z^I u(0,\cdot)\| + CM\varepsilon \int_0^t (1+\tau)^{\frac{1-n}{2}} M_s(\tau)\, d\tau, \quad |I| \le s,$$

for the exponential factor in (6.3.6) will be close to 1 by (6.5.9). We can write $Z^I\partial_j$ as a linear combination of $\partial_k Z^J$ with $|J| \le |I|$ by an argument used a moment ago, so it follows that

$$M_s(t) \le C\big(M_s(0) + M\varepsilon \int_0^t (1+\tau)^{\frac{1-n}{2}} M_s(\tau)\, d\tau\big). \tag{6.5.10}$$

In view of Gronwall's lemma it follows that

$$M_s(t) \le CM_s(0)\exp\Big(\int_0^t CM\varepsilon(1+\tau)^{\frac{1-n}{2}}\,d\tau\Big). \tag{6.5.11}$$

Choose M so that $3CM_s(0) \le M\varepsilon$. Then we have attained that (6.5.6) holds for small T and for any T implies the same estimate with M replaced by $M/2$ if ε is small enough. Thus (6.5.6) follows by continuous induction for every T such that the Cauchy problem has a solution for $0 \le t \le T$, so this set is closed. Since it is open by Theorem 6.4.11 the global existence follows.

Remark. The proof works if we just prescribe small Cauchy data such that $Z^I u'(0,\cdot) \in L^2$ for $|I| \le s$ for some $s \ge 2\kappa$, and $\|Z^I u'(t,\cdot)\|$ is then bounded when $|I| \le s$. If the support is compact this follows at once from the preceding proof. Otherwise one can choose $\chi \in C_0^\infty(\mathbf{R}^n)$ equal to 1 in a neighborhood of 0 and note that the bounds for the solution with Cauchy data $u_j\chi(\delta\cdot)$ are independent of δ as $\delta \to 0$.

If $n \le 3$ the proof of Theorem 6.5.2 does not break down completely. Firstly, the existence proof remains valid if the perturbation vanishes of sufficiently high order and $n > 1$. Assume that $\gamma^{jk}(u')$ and $f(u')$ vanish of order p and $p+1$ respectively at 0. Then we get at least p factors for which we can use the estimate (6.5.7), and it follows that there is global existence for small ε if $p(n-1) > 2$. When $n = 3$ it is therefore only the quadratic terms that can cause difficulties. Secondly, even if the perturbation is just of second order we get an estimate for the lifespan of the solution by just looking carefully at the exponential factors which appear in (6.3.6) and in the application of Gronwall's lemma. Assume that for some small $\delta > 0$ we have

$$\varepsilon\int_0^T (1+\tau)^{\frac{1-n}{2}}\,d\tau \le \delta. \tag{6.5.12}$$

Then (6.5.10) is modified by a factor $e^{C\delta}$ in the right-hand side, and (6.5.11) is replaced by

$$M_s(t) \le Ce^{C\delta}M_s(0)\exp(CMe^{C\delta}\delta).$$

Choose M as before so that $3CM_s(0) \le M\varepsilon$. When δ is so small that

$$e^{C\delta}\exp(CMe^{C\delta}\delta) < 3/2,$$

we conclude that (6.5.6) holds for small T and always implies the same estimate with M replaced by $M/2$, if T satisfies (6.5.12) and ε is small. Hence we have proved

Theorem 6.5.3. *The Cauchy problem* (6.5.1), (6.5.2) *with* $u_j \in C_0^\infty(\mathbf{R}^n)$ *has for small* ε *a solution for* $0 \le t \le T_\varepsilon$ *if*

$$\varepsilon\int_0^{T_\varepsilon} (1+t)^{\frac{1-n}{2}}\,dt = c,$$

where $c > 0$ *depends on* u_0, u_1. *Equivalently, with some other* c, *a solution exists for* $0 \le t \le T_\varepsilon$ *where for some* $c > 0$

$$T_\varepsilon = \begin{cases} e^{c/\varepsilon}, & \text{if } n = 3,\\ c/\varepsilon^2, & \text{if } n = 2,\\ c/\varepsilon, & \text{if } n = 1.\end{cases}$$

If the perturbation vanishes of order $p+1$ as above and $p(n-1) \le 2$, then we get existence for $0 \le t \le T_\varepsilon$ if

$$\varepsilon^p\int_0^{T_\varepsilon} (1+t)^{p(1-n)/2}dt \le c$$

for a sufficiently small constant c. When $n = 2$ this gives "almost global existence" if $p = 2$. However, the most interesting problem is to determine how quadratic perturbations influence the lifespan T_ε. When $n = 1$ we know this from (6.1.5). We shall now prove that for $n = 2$ or $n = 3$ there is at least a similar *lower* bound for T_ε. The main point is that one can construct an approximate solution with an error which is small compared to ε as long as it exists. This will make the exponential factors controlled by (6.5.12) in the preceding proof harmless even if δ is not small. For the sake of brevity we assume that $f = 0$. The extension to general f was prepared in Lemma 2.3.1 though, and it will be carried out at the end of the section.

Assuming at first that $n = 3$ and that $u_0, u_1 \in C_0^\infty(\mathbf{R}^3)$, we shall look for an approximate solution to (6.5.1), (6.5.2) of the form

$$u(t, r\omega) = \varepsilon r^{-1} U(\omega, \varepsilon \log t, r - t), \quad |\omega| = 1, r > 0.$$

This is motivated by the asymptotic formula $\varepsilon r^{-1} F_0(\omega, r - t)$ for the solution of the unperturbed wave equation given in (6.2.6) and the hint from Theorem 6.5.3 that nonlinear effects start to be important when $\varepsilon \log t$ attains a certain value essentially independent of ε. Since

$$\Box u = r^{-1}((\partial_t - \partial_r)(\partial_t + \partial_r) - r^{-2}\Delta_\omega) ru,$$

with Δ_ω denoting the Laplacian in S^2, the main term in $\Box u$ is obtained when $\partial_t + \partial_r$ acts on the argument $s = \varepsilon \log t$ and $\partial_t - \partial_r$ acts on $q = r - t$, which gives

$$-2\varepsilon^2 (tr)^{-1} U''_{sq}(\omega, s, q).$$

Writing as in the introduction

$$g^{jk}(u') = g^{jk}(0) + \sum_{l=0}^{n} g^{jkl} \partial_l u + O(|u'|^2), \tag{6.5.13}$$

we find that the main nonlinear terms in the equation (6.5.1) are

$$\varepsilon^2 r^{-2} G(\omega) U'_q U''_{qq},$$

where

$$G(\omega) = \sum_{j,k,l=0}^{n} g^{jkl} \hat\omega_j \hat\omega_k \hat\omega_l; \quad \hat\omega = (-1, \omega_1, \omega_2, \omega_3) = q'. \tag{6.5.14}$$

Thus it is natural to choose U so that

$$2U''_{sq}(\omega, s, q) = G(\omega) U'_q(\omega, s, q) U''_{qq}(\omega, s, q).$$

If U vanishes for large q this is equivalent to

$$4\partial U(\omega, s, q)/\partial s = G(\omega)(\partial U(\omega, s, q)/\partial q)^2. \tag{6.5.15}$$

When t is large but $\varepsilon \log t$ is still small, the nonlinear effects should not yet be important, so it is natural to require the initial condition

$$U(\omega, 0, q) = F_0(\omega, q), \tag{6.5.16}$$

where F_0 is the Friedlander radiation field in (6.2.6). This Cauchy problem is easy to solve:

Lemma 6.5.4. *The Cauchy problem* (6.5.15), (6.5.16) *has a unique* C^∞ *solution for* $0 \le s < A$ *where*

$$A = (\max_{\omega,\varrho} \tfrac{1}{2}G(\omega)\partial^2 F_0(\omega,\varrho)/\partial\varrho^2)^{-1}, \tag{6.5.17}$$

but the second order derivatives are unbounded when $s \to A$ *if* $A < \infty$, *which is true unless* $G \equiv 0$ *or* $(u_0, u_1) \equiv 0$.

Proof. If we set $u = \partial U/\partial q$ the equation (6.5.15) implies that u satisfies Burgers' equation with parameters

$$2\partial u(\omega,s,q)/\partial s = G(\omega)u(\omega,s,q)\partial u(\omega,s,q)/\partial q. \tag{6.5.15$'$}$$

We have $\int u(\omega,s,q)\,dq = 0$ for all (ω,s) since this is true when $s = 0$, and $|q| \le M$ when $(\omega,s,q) \in \operatorname{supp} u$ if $|y| \le M$ when $y \in \operatorname{supp} u_0 \cup \operatorname{supp} u_1$. Thus $\partial U/\partial q = u$ for a unique U with such support, and U satisfies (6.5.15). The lemma is now a consequence of the discussion of the lifespan of solutions of Burgers' equation in Section 2.3; it applies with no change when the parameters ω are present. (The lemma could also have been proved using the Hamilton-Jacobi integration theory.) That $A < \infty$ except in the trivial cases listed follows from Theorem 6.2.2, for the second order derivative of a function of compact support which is not identically 0 takes both positive and negative values.

We have now found a good approximation when t is fairly large, but it does not have the correct Cauchy data so we piece it together with the solution εw_0 of the homogeneous wave equation with Cauchy data (6.5.2). To do so we choose $\chi \in C^\infty(\mathbf{R})$ decreasing, equal to 1 in $(-\infty,1)$ and equal to 0 in $(2,\infty)$, and we set for $0 \le \varepsilon t < e^{A/\varepsilon}$

$$w_\varepsilon(t,x) = w(t,x) = \varepsilon\big(\chi(\varepsilon t)w_0(t,x) + (1-\chi(\varepsilon t))r^{-1}U(\omega,\varepsilon\log(\varepsilon t), r-t)\big). \tag{6.5.18}$$

Thus we shift when $1 \le \varepsilon t \le 2$ from the solution of the homogeneous wave equation to the approximation constructed for large t. The initial conditions (6.5.2) are of course satisfied by w, and we shall now estimate how well (6.5.1) is fulfilled (with $f = 0$).

Lemma 6.5.5. *With* w *defined by* (6.5.18) *and*

$$R = \sum_{j,k=0}^{n} g^{jk}(w')\partial_j\partial_k w, \tag{6.5.19}$$

we have $R, w \in C^\infty$ *when* $t < e^{A/\varepsilon}$, *where* A *is defined by* (6.5.17). *If* $(t,x) \in \operatorname{supp} w$ *we have*

$$||x| - t| \le \sup\{|y|; y \in \operatorname{supp} u_0 \cup \operatorname{supp} u_1\}.$$

For fixed $B \in (0,A)$ *we have for small* ε *and all* I *if* $\varepsilon\log t \le B$

$$|Z^I w(t,x)| \le C_{I,B}\varepsilon(1+t)^{-1}, \tag{6.5.20}$$

$$|Z^I R(t,x)| \le C_{I,B}\varepsilon^2(1+t)^{-2}(1+\varepsilon t)^{-1}. \tag{6.5.21}$$

Proof. If w is replaced by the solution εw_0 of the wave equation with initial data εu_0, εu_1, then (6.5.20) follows from (6.2.10). To prove (6.5.20) when $\varepsilon t \ge 1$ and $\varepsilon\log t \le B$ we note that $Z^I\chi(\varepsilon t)$ is uniformly bounded for any I when $|x| < t + C$, that $Z\log(\varepsilon t)$ is homogeneous of degree ≤ 0, and that Zq is either homogeneous of degree 0 or else equal to $-\omega_j q$, if $Z = x_j\partial_t + t\partial_j$. This implies that Z^I applied to the second term in (6.5.18) is a sum with bounded coefficients of derivatives of U multiplied by functions homogeneous of degree ≤ -1 and powers of $q = r - t$, which is bounded in the support. This proves (6.5.20).

To prove (6.5.21) we distinguish three different cases:

i) When $\varepsilon t \le 1$ we have $w(t,x) = \varepsilon w_0(t,x)$, and

$$R = \sum_{j,k} (g^{jk}(w') - g^{jk}(0))\partial_j\partial_k w$$

since $\Box w_0 = 0$. Hence (6.5.21) follows from (6.5.20), for the factor $1+\varepsilon t$ plays no role.

ii) Now consider the transition zone where $1 \le \varepsilon t \le 2$. In addition to the arguments in case i) we must then also estimate

$$\begin{aligned}\Box w &= \Box(w - \varepsilon w_0)\\ &= \varepsilon\big(((1-\chi(\varepsilon t))\Box - 2\varepsilon\chi'(\varepsilon t)\partial_t - \varepsilon^2\chi''(\varepsilon t)\big)(r^{-1}U(\omega, \varepsilon\log(\varepsilon t), r-t) - w_0(t,x))\big).\end{aligned}$$

In the term where χ is differentiated twice the desired bound $O(\varepsilon^4)$ is immediately clear. In the term where χ is differentiated once we use that

$$r^{-1}(U(\omega,\varepsilon\log(\varepsilon t), r-t) - F_0(\omega, r-t)), \quad r^{-1}F_0(\omega, r-t) - w_0(t,x)$$

are $O(\varepsilon r^{-1})$ and $O(r^{-2})$ respectively, and that such bounds still hold after multiplication by any Z^I. In the first case this follows from the proof of (6.5.20) since $0 \le \log(\varepsilon t) \le \log 2$; in the second case it follows from (6.2.11). What remains is to study

$$\varepsilon(1-\chi(\varepsilon t))r^{-1}((\partial_t - \partial_r)(\partial_t + \partial_r) - r^{-2}\Delta_\omega)U(\omega, \varepsilon\log(\varepsilon t), r-t).$$

Here $\partial_t + \partial_r$ must act on $\varepsilon\log(\varepsilon t)$, producing a factor ε/t, which gives the desired bound if we recall the proof of (6.5.20) once more.

iii) Let $2/\varepsilon \le t \le e^{B/\varepsilon}$. Then

$$\Box w = \varepsilon r^{-1}((\partial_t - \partial_r)(\partial_t + \partial_r) - r^{-2}\Delta_\omega)U(\omega, \varepsilon\log(\varepsilon t), r-t),$$

and as just observed $\partial_t + \partial_r$ must act on $\varepsilon\log(\varepsilon t)$, which yields a factor ε/t. Writing $s = \varepsilon\log(\varepsilon t)$ and $q = r-t$, we obtain

$$|\Box w + 2\varepsilon^2 r^{-1}t^{-1}U''_{sq}(\omega,s,q)| \le C\varepsilon t^{-3}.$$

With the notation in (6.5.14) we have

$$|\partial^\alpha w - \varepsilon r^{-1}\hat\omega^\alpha\partial_q^{|\alpha|}U(\omega,s,q)| \le C\varepsilon r^{-2},$$

$$\Big|\sum_{j,k=0}^n (g^{jk}(w') - g^{jk}(0))\partial_j\partial_k w - \varepsilon^2 r^{-2}G(\omega)U'_qU''_{qq}\Big| \le C\varepsilon^2 r^{-3}.$$

Recalling that $2U''_{sq} = G(\omega)U'_qU''_{qq}$ we conclude that $|R| \le C\varepsilon t^{-3}$, for $1/t - 1/r = O(1/t^2)$. This proves (6.5.21) when $I = 0$, and using (6.5.20) we obtain (6.5.21) for arbitrary I too.

We shall write the solution u of (6.5.1) (with $f=0$) and (6.5.2) in the form $u = v + w$ where w is the approximate solution studied in Lemma 6.5.5. Then the Cauchy problem is restated as

(6.5.1)′ $$\sum_{j,k=0}^n g^{jk}(v'+w')\partial_j\partial_k v + R + \sum_{j,k=0}^n (g^{jk}(v'+w') - g^{jk}(w'))\partial_j\partial_k w = 0$$

(6.5.2)′ $$v = \partial_t v = 0 \quad \text{when } t = 0.$$

Here R is defined by (6.5.19).

Lemma 6.5.6. *Assume that (6.5.1)′, (6.5.2)′ has a C^∞ solution for $0 \le t \le T$ where $\varepsilon \log T \le B < A$, with A defined by (6.5.17). If $0 < \varepsilon < \delta_B$, it follows that*

$$\|Z^I v'(t,\cdot)\| \le C_{I,B}\varepsilon^2 \log(1/\varepsilon), \quad 0 \le t \le T, \tag{6.5.22}$$

where δ_B and $C_{I,B}$ are independent of T and ε.

Proof. The proof is parallel to that of Theorem 6.5.2. We shall estimate $Z^I v'$ by applying the standard energy estimate (6.3.6) to the equation obtained when (6.5.1)′ is multiplied by Z^I. By Lemma 6.5.5 we have

$$\|Z^I R(t,\cdot)\| \le C_{I,B}\varepsilon^2(1+t)^{-1}(1+\varepsilon t)^{-1}, \tag{6.5.23}$$

for the measure of the support is $O(1+t)^2$, and we shall use that

$$\int_0^\infty (1+t)^{-1}(1+\varepsilon t)^{-1} dt = (\varepsilon-1)^{-1}\log\varepsilon. \tag{6.5.24}$$

From (6.5.1)′ and (6.5.2)′ we also find that $Z^I v = O(\varepsilon^2)$ for every I when $t = 0$.
Choose s and N satisfying (6.5.5) and assume that

$$N_s(t) = \sum_{|I|\le s} \|Z^I v'(t,\cdot)\| \le \varepsilon, \quad \text{if } 0 \le t \le T, \tag{6.5.25}$$

where the equality is of course a definition. If we prove (6.5.22) under this assumption then we see that for small ε we actually have (6.5.25) with ε replaced by $\varepsilon/2$, so (6.5.25) and (6.5.22) will follow by "continuous induction". By Proposition 6.5.1 it follows from (6.5.25) that

$$|Z^I v'(t,\cdot)| \le CN_s(t)(1+t)^{-1} \le C\varepsilon(1+t)^{-1}, \quad 0 \le t \le T,\ |I| \le N. \tag{6.5.26}$$

Combining (6.5.20) and (6.5.25), (6.5.26) we obtain for some constant C'

$$\begin{aligned} &\|Z^I u'(t,\cdot)\| \le C'\varepsilon, && 0 \le t \le T,\ |I| \le s,\\ &|Z^I u'(t,\cdot)| \le C'\varepsilon(1+t)^{-1}, && 0 \le t \le T,\ |I| \le N. \end{aligned} \tag{6.5.27}$$

When $|I| \le s$ we are now ready to estimate the right-hand side of the equation for $Z^I v$ obtained from (6.5.1)′

$$\sum_{j,k=0}^{n} g^{jk}(u')\partial_j\partial_k Z^I v = \sum_0^4 R_j,$$

$$R_0 = -Z^I R, \quad R_1 = [\Box, Z^I]v, \quad R_2 = \sum [\gamma^{jk}(u'), Z^I]\partial_j\partial_k v,$$

$$R_3 = \sum \gamma^{jk}(u')[\partial_j\partial_k, Z^I]v, \quad R_4 = -Z^I \sum (g^{jk}(v'+w') - g^{jk}(w'))\partial_j\partial_k w.$$

As before the hardest term is R_2, which is a sum of derivatives of γ^{jk} multiplied by components of

$$Z^{J_1}u' \cdots Z^{J_r}u' Z^K \partial_j\partial_k v$$

where $|J_1| + \cdots + |J_r| + |K| \le s$, $J_i \ne 0$ for every i, and $r \ne 0$. Estimating the last factor using (6.5.25) or (6.5.26) and the others by means of (6.5.27) we obtain

$$\|R_2(t,\cdot)\| \le C\varepsilon(1+t)^{-1}N_s(t).$$

For R_3 and R_4 the same estimate is obtained even more easily if we use Taylor's formula to write $\gamma^{jk}(u')$ and $g^{jk}(v'+w')-g^{jk}(w')$ as scalar products with u' and v' respectively. After writing

$$R_1 = \sum_{|J|<|I|} c_{I,J} Z^J \Box v$$

and substituting the expression for $\Box v$ given by (6.5.1)′, we have the same estimate for R_1 too, and R_0 was estimated in (6.5.23), (6.5.24). Now it follows from (6.5.27) that with the notation in (6.3.6) we have

$$\int_0^T |\gamma'(t)|\,dt \le C\varepsilon \int_0^T dt/(1+t) = C\varepsilon \log(1+T) \le 2CB, \quad \text{if } \varepsilon \log T \le B.$$

Recalling that the Cauchy data of $Z^I v$ are $O(\varepsilon^2)$, we obtain using (6.3.6)

$$\|\partial Z^I v(t,\cdot)\| \le C\big(\varepsilon^2 \log(1/\varepsilon) + \int_0^t \varepsilon N_s(\tau)\,d\tau/(1+\tau)\big).$$

This implies with another C, depending on B, that

$$N_s(t) \le C(\varepsilon^2 \log(1/\varepsilon) + \int_0^t \varepsilon N_s(\tau)\,d\tau/(1+\tau)),$$

hence by Gronwall's lemma

$$N_s(t) \le C\varepsilon^2 \log(1/\varepsilon) \exp\Big(\int_0^t C\varepsilon\,d\tau/(1+\tau)\Big) \le C\varepsilon^2 \log(1/\varepsilon)\exp(2CB).$$

This completes the proof of (6.5.22).

From Lemma 6.5.6 and the local existence theorem it follows at once that for small ε the Cauchy problem (6.5.1)′, (6.5.2)′ has a solution satisfying (6.5.22) when $t \le \exp(B/\varepsilon)$, if $B < A$. By Proposition 6.5.1

$$|u'(t,x) - w'(t,x)| = |v'(t,x)| \le C(1+t)^{-1}(1+||x|-|t||)^{-\frac12}\varepsilon^2 \log(1/\varepsilon), \quad 0 \le t \le e^{B/\varepsilon}.$$

Since $v(t,x) = 0$ when $|x|-|t| >$ constant it follows that

$$\varepsilon^{-1}(1+t)|u(t,x) - w(t,x)| \le C(1+||x|-|t||)^{\frac12}\varepsilon \log(1/\varepsilon), \quad 0 \le t \le e^{B/\varepsilon}.$$

When $t = e^{s/\varepsilon}$ and $x = (t+q)\omega$, $|\omega| = 1$, we have

$$tw(t,x) = \varepsilon U(\omega, s+\varepsilon \log \varepsilon, q)t/(t+q)$$

when $\varepsilon e^{s/\varepsilon} > 2$, so we have proved:

Theorem 6.5.7. *The Cauchy problem* (6.5.1) *(with $f = 0$),* (6.5.2) *with $u_j \in C_0^\infty(\mathbf{R}^3)$ has a C^∞ solution u_ε for $0 \le t < T_\varepsilon$ where*

$$\underline{\lim}_{\varepsilon\to 0}\, \varepsilon \log T_\varepsilon \ge A = (\max_{\omega,\varrho} \tfrac12 G(\omega)\partial^2 F_0(\omega,\varrho)/\partial\varrho^2)^{-1}. \tag{6.5.28}$$

Here $\omega \in S^2$ and G is defined in (6.5.14) *while εF_0 is the Friedlander radiation field of the solution of the Cauchy problem for the unperturbed equation. If U is the solution of* (6.5.15), (6.5.16), *then*

$$\varepsilon^{-1} e^{s/\varepsilon} u_\varepsilon(e^{s/\varepsilon}, (e^{s/\varepsilon}+\varrho)\omega) - U(\omega,s,\varrho) \to 0, \quad \text{as } \varepsilon \to 0, \tag{6.5.29}$$

locally uniformly in $S^2 \times (0, A) \times \mathbf{R}$; *in fact, the difference is locally uniformly* $O(\varepsilon \log \varepsilon)$.

When g^{jk} only depend on $\partial u/\partial t$ and the Cauchy data are rotationally symmetric, then John [2] has proved that for the lifespan T_ε of the solution one can replace $\underline{\lim}$ in (6.5.28) by lim and inequality by equality. The idea of the proof is that in polar coordinates one actually has a problem with just one space variable where one can use a modification of the proof of Theorem 4.3.1. For the details of the proof we refer to John [2] or Hörmander [1]. The latter paper also contains Theorem 6.5.7, which was proved independently by John [3] with a different argument. Theorem 6.5.2 was proved by Klainerman [1] when $n \geq 6$ with a far more complicated proof. The proof given here is essentially that of Klainerman [3] who proved the theorem for $n > 4$ and also proved Theorem 6.5.3 which had been established somewhat earlier for the critical dimension $n = 3$ by John and Klainerman [1]. In Hörmander [1] the analogue of Theorem 6.5.7 for $n = 2$ was also proved with essentially the same arguments. The only additional complication is that the Friedlander radiation field does not have compact support in the radial variable, but that is compensated by its symbol properties. Thus we have the following result also:

Theorem 6.5.8. *The Cauchy problem* (6.5.1) *(with* $f = 0$*),* (6.5.2) *with* $u_j \in C_0^\infty(\mathbf{R}^2)$ *has a* C^∞ *solution* u_ε *for* $0 \leq t < T_\varepsilon$ *where*

$$\underline{\lim}_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} \geq A = (\max_{\omega, \varrho} G(\omega) \partial^2 F_0(\omega, \varrho)/\partial \varrho^2)^{-1}. \tag{6.5.30}$$

Here $\omega \in S^1$ *and* G *is defined by the analogue of* (6.5.14), *while* εF_0 *is the Friedlander radiation field of the solution of the Cauchy problem for the unperturbed wave equation. If* U *is the solution of* (6.5.15), (6.5.16) *then*

$$s\varepsilon^{-2} u_\varepsilon(s^2/\varepsilon^2, (s^2/\varepsilon^2 + \varrho)\omega) - U(\omega, s, \varrho) \to 0, \quad \text{as } \varepsilon \to 0, \tag{6.5.31}$$

locally uniformly in $S^1 \times (0, A) \times \mathbf{R}$; *in fact, the difference is locally uniformly* $O(\varepsilon^{\frac{1}{2}})$.

When $G(\omega) \equiv 0$ Theorems 6.5.7 and 6.5.8 suggest a much better order of magnitude for the lifespan. In fact, Christodoulou [1] and Klainerman [4] have then proved global existence for small ε when $n = 3$. This will be done in Sections 6.6 and 6.7 here.

We shall now outline an extension of Theorem 6.5.7 to the case where the function f in (6.5.1) is not identically 0. In analogy with (6.5.13), (6.5.14) we introduce

$$f(u') = \sum_{j,k} f^{jk} \partial_j u \partial_k u + O(|u'|^3), \tag{6.5.13$'$}$$

$$F(\omega) = \sum_{j,k} f^{jk} \hat{\omega}_j \hat{\omega}_k; \quad \hat{\omega} = (-1, \omega_1, \omega_2, \omega_3). \tag{6.5.14$'$}$$

We get new nonlinear terms dominated by $F(\omega) U_q'^2$, and with the notation $u = U_q'$ as in the proof of Lemma 6.5.4 this gives instead of (6.5.15)$'$ the modified Burgers' equation

$$2\partial u(\omega, s, q)/\partial s = G(\omega) u(\omega, s, q) \partial u(\omega, s, q)/\partial q - F(\omega) u(\omega, s, q)^2, \tag{6.5.32}$$

which is of the form studied in Lemma 2.3.1. Again we solve this equation with the initial condition $u(\omega, 0, q) = \partial F_0(\omega, q)/\partial q$. The solution exists for $0 \leq s < A$ where (see Lemma 2.3.1)

$$A = \big(\max \tfrac{1}{2}(G(\omega) \partial^2 F_0(\omega, \varrho)/\partial \varrho^2 - F(\omega) \partial F_0(\omega, \varrho)/\partial \varrho)\big)^{-1} \tag{6.5.33}$$

is finite unless $(G, F) \equiv 0$ or $(u_0, u_1) \equiv 0$. It is no longer true that $\int u(\omega, s, q)\, dq$ vanishes for $s \neq 0$; in fact,

$$2\frac{\partial}{\partial s}\int u(\omega, s, q)\, dq = -\int F(\omega)u(\omega, s, q)^2 dq,$$

which is hardly ever zero. We define $U(\omega, s, q)$ as the solution of $\partial U/\partial q = u$ which vanishes for $q > M$ if M is an upper bound for $|y|$ when $y \in \operatorname{supp} u_0 \cup \operatorname{supp} u_1$. When $q < -M$ it is a function $U_-(\omega, s)$ independent of q, and we have

$$2U''_{sq} = G(\omega)U'_q U''_{qq} - F(\omega)U'^2_q. \tag{6.5.34}$$

To avoid singularities when $r = 0$ we must cut off by taking a function $\psi(t, x)$ in C^∞ which is homogeneous of degree 0, equal to 1 in a conic neighborhood of the light cone and equal to 0 in a conic neighborhood of the t axis. We keep the definition (6.5.18) of w with U replaced by ψU. When $q < -M$ this is equal to $\psi(t, x)U_-(\omega, \varepsilon\log(\varepsilon t))$, so we have

$$|\partial^\alpha(\psi(t, x)U_-(\omega, \varepsilon\log(\varepsilon t)))| \le C_{\alpha,B}(1+t)^{-|\alpha|} \quad \text{if } \varepsilon t \ge 1,\ \varepsilon\log t \le B.$$

This proves that (6.5.20) remains valid. When $q < -M$ we find that

$$|Z^I \Box w| \le C_{I,B}\varepsilon(1+t)^{-3} \quad \text{if } \varepsilon t \ge 1,\ \varepsilon\log t \le B,$$

so (6.5.21) remains valid too, with

$$R = \sum g^{jk}(w')\partial_j\partial_k w - f(w').$$

However, the measure of the support of R is now only $O(1+t)^3$, so (6.5.23) is replaced by

$$\|Z^I R(t, \cdot)\| \le C_{I,B}\varepsilon t^{-3/2} \quad \text{if } \varepsilon t \ge 1,\ \varepsilon\log t \le B,$$

and the integral of this from $1/\varepsilon$ to ∞ is $O(\varepsilon^{3/2})$. In (6.5.1)′ we now obtain another term $f(v' + w') - f(w')$ in the right-hand side, which gives another error term

$$R_5 = Z^I(f(v' + w') - f(w'))$$

in the subsequent argument. It is estimated just as the others after writing $f(v' + w') - f(w'))$ as a scalar product with v', so we obtain

$$N_s(t) \le C\varepsilon^{3/2}, \quad 0 \le t \le e^{B/\varepsilon}.$$

Hence we have proved that

Theorem 6.5.9. *The Cauchy problem* (6.5.1), (6.5.2) *with* $u_j \in C_0^\infty(\mathbf{R}^3)$ *has a* C^∞ *solution* u_ε *for* $0 \le t < T_\varepsilon$ *where*

$$\varliminf_{\varepsilon\to 0} \varepsilon\log T_\varepsilon \ge A$$

with A *now defined by* (6.5.33). *Here* $\omega \in S^2$ *and* F, G *are defined by* (6.5.14), (6.5.14)′ *while* εF_0 *is the Friedlander radiation field of the solution of the Cauchy problem for the unperturbed wave equation. If* U *is the solution of* (6.5.34), (6.5.16) *vanishing for* $q \gg 0$ *then* (6.5.29) *holds locally uniformly in* $S^2 \times (0, A) \times \mathbf{R}$*; in fact, the difference is locally uniformly* $O(\varepsilon^{\frac{1}{2}})$.

Remark 1. Theorem 6.5.9 was extended in Hörmander [7] to fully non-linear perturbations of the wave equation

$$\Box u = f(u', u'') \tag{6.5.35}$$

where f vanishes of second order at $(0,0)$. The new problem posed by this generalization of (6.5.1) is that (6.5.32) is replaced by a more general equation of the form

$$\partial u/\partial s = a(\omega)(\partial u/\partial q)^2 + 2b(\omega)u\partial u/\partial q + c(\omega)u^2.$$

The lifespan of solutions of the Cauchy problem for this equation can still be determined explicitly, but the formula for it is fairly complicated so the expression for A in the extension of Theorem 6.5.9 is somewhat involved.

Remark 2. If one wants to apply the methods of this section to the equation

$$\square u = f(u, u', u''), \tag{6.5.36}$$

where f vanishes of second order at $(0,0,0)$ but may depend on u, there is another difficulty. The energy integral method gives naturally only an estimate of $\|u'(t,\cdot)\|$, and one must expect to lose a factor t when passing to an estimate for $\|u(t,\cdot)\|$. For $n > 5$ this will suffice to prove a global existence theorem for small ε using the proof of Theorem 6.5.2, and the result was extended to $n = 5$ by Li and Chen [1]. In Hörmander [8] it was proved that if there is no u^2 term, that is, $f(u,0,0) = O(u^3)$, then (6.5.36), (6.5.2) has a global solution for small ε when $n \geq 4$, and that the lifespan T_ε is $\geq e^{c/\sqrt{\varepsilon}}$ for some $\varepsilon > 0$ when $n = 3$; this was improved to $T_\varepsilon \geq e^{c/\varepsilon}$ by Lindblad [2]. For arbitrary f it was also proved by Hörmander [8] that $T_\varepsilon \geq e^{c/\varepsilon}$ for some $c > 0$ when $n = 4$; this was improved to $T_\varepsilon \geq e^{c/\varepsilon^2}$ by Li and Zhou [1]. When $n = 3$ it was already proved by John [4] for the equation $\square u + u^2 = 0$ that $\varepsilon^2 T_\varepsilon$ lies between two positive bounds for small ε. Lindblad [1] proved that the limit as $\varepsilon \to 0$ exists and gave a description of it.

Remark 3. Even if it is hard to doubt that (6.5.28) and (6.5.30) always give the precise asymptotic lifespan of the solutions there are no proofs except for the rotationally symmetric three dimensional case studied by F. John [2] and the recent study by Alinhac [3] of some very special equations and data with two space dimensions. Theorems 6.5.7 and 6.5.8 only show that if the lifespan should be much longer then the asymptotics would be of a different kind. However, Alinhac [1, 2] has given a more precise lower bound for the lifespan in the case discussed in Theorem 6.5.7 and proved very refined results on the asymptotic behavior close to this time which reinforce the belief that Theorem 6.5.7 is fairly precise. (See also the bibliographies in these references.)

6.6. The null condition in three dimensions. In the main results of this section the number n of space variables will be equal to 3, but at first we allow any $n \geq 3$. We shall consider a quasilinear second order equation

$$\sum_{j,k=0}^{n} g^{jk}(u,u')\partial_j\partial_k u = f(u,u') \tag{6.6.1}$$

with small Cauchy data

$$u(0,x) = \varepsilon u_0(x), \quad \partial_0 u(0,x) = \varepsilon u_1(x), \tag{6.6.2}$$

where $u_j \in C_0^\infty(\mathbf{R}^n)$. We shall often write $t = x_0$ and $\vec{x} = (x_1, \dots, x_n)$. We assume that $u \equiv 0$ is a solution of (6.6.1) and that the linearisation of (6.6.1) there is the wave equation, that is,

$$\sum g^{jk}(0,0)\partial_j\partial_k = \square, \quad f = df = 0 \text{ at } (0,0).$$

Definition 6.6.1. The equation (6.6.1) is said to satisfy the *null condition* if

$$g^{jk}(u,u') = \sum_{l=0}^{n} g^{jkl}\partial_l u + O(|u|+|u'|)^2,$$

$$f(u,u') = \sum_{j,k=0}^{n} f^{jk}\partial_j u\partial_k u + O(|u|+|u'|)^3,$$

where

$$\sum_{j,k,l=0}^{n} g^{jkl}\xi_j\xi_k\xi_l = 0, \quad \sum_{j,k=0}^{n} f^{jk}\xi_j\xi_k = 0 \text{ when } \xi_0^2-\xi_1^2-\cdots-\xi_n^2=0. \tag{6.6.3}$$

Thus the null condition requires that the quadratic terms are independent of u and that with the notation used in Theorem 6.5.9 we have $G(\omega)\equiv 0$ and $F(\omega)\equiv 0$. We shall improve Theorem 6.5.9 when $A=\infty$ to the following result of Christodoulou [1] and Klainerman [4] (see also Hörmander [6]):

Theorem 6.6.2. *If* (6.6.1) *satisfies the null condition,* $n=3$, $u_j\in C_0^\infty(\mathbf{R}^3)$, *and* ε *is sufficiently small, then the Cauchy problem* (6.6.1), (6.6.2) *has a* C^∞ *solution for* $t\ge 0$.

The proof will occupy the entire section. The main change of the methods used in Section 6.5 is that we shall use energy estimates of the form (6.3.14). To do so we must examine the relation of the null condition to the vector fields (6.2.8), (6.2.9) and the constant vector fields. These suggest using the following norm on covectors

$$|\xi|_x = \Big(\sum_{j<k} Z_{jk}(x,\xi)^2 + Z_0(x,\xi)^2 + \sum_0^n \xi_j^2\Big)^{\frac12} = (|A'(x)\wedge\xi|^2/4 + \langle x,\xi\rangle^2 + |\xi|^2)^{\frac12}, \tag{6.6.4}$$

where the norms are Euclidean and $A(x)=x_0^2-x_1^2-\cdots-x_n^2$. The following lemma is perhaps clarified by allowing A to be any nondegenerate real quadratic form.

Lemma 6.6.3. *Let* G *be a* k *linear form on* $\mathbf{R}^{1+n}$. *Then*

$$|G(\xi^1,\dots,\xi^k)| \le C(1+|x|)^{-1}|\xi^1|_x\cdots|\xi^k|_x; \quad \xi^j, x\in\mathbf{R}^{1+n}; \tag{6.6.5}$$

if and only if, with B *denoting the dual quadratic form of* A,

$$G(\xi,\dots,\xi)=0 \quad \text{when } \xi\in\mathbf{R}^{1+n},\ B(\xi)=0. \tag{6.6.6}$$

Proof. Let $\xi^1=\cdots=\xi^k=A'(x)$ where $A(x)=0$ but $x\ne 0$. Then it follows from (6.6.4) that $|\xi^j|_{tx}=|A'(x)|$, so the right-hand side of (6.6.5) with x replaced by tx is $O(1/t)$ as $t\to\infty$. This proves that (6.6.5) implies (6.6.6). Now assume that (6.6.6) holds, and let $|\xi^j|_x=1$ for all j. Then $|\xi^j|\le 1$ so (6.6.5) is obvious if $|x|\le 1$. Write

$$\xi^j = r^j + t_jA'(x)$$

where r^j is orthogonal to $A'(x)$ in the Euclidean sense. Then

$$|\xi^j\wedge A'(x)| = |r^j\wedge A'(x)| = |r^j||A'(x)|,$$

so we have a bound for $|r^j|(|x|+1) + |t_j A'(x)| + |t_j A(x)|$. It follows that all terms in the expansion of $G(\xi^1, \dots, \xi^k)$ containing some r^j can be estimated by $C/(1+|x|)$. It remains to estimate

$$t_1 \cdots t_k G(A'(x), \dots, A'(x))$$

when $|x| > 1$. By (6.6.6) we have $|G(\xi, \dots, \xi)| \le C|\xi|^{k-2}|B(\xi)|$, hence

$$|G(A'(x), \dots, A'(x))| \le C|A'(x)|^{k-2}|B(A'(x))| = 4C|A'(x)|^{k-2}|A(x)|.$$

Since $t_j A(x)$ and $t_j|A'(x)|$ have fixed bounds we obtain

$$|t_1 \cdots t_k G(A'(x), \dots, A'(x))| \le 4C|t_1 \cdots t_k||A(x)||A'(x)|^{k-2} \le C'/|x|,$$

which completes the proof.

We shall use Lemma 6.6.3 for $k = 2$ and for $k = 3$. For $k = 3$ a closely related estimate of the quadratic term in (6.6.1) is also required:

Lemma 6.6.4. *If g^{jk} satisfy the null condition* (6.6.3), *then*

$$\Big|\sum_{j,k,l=0}^{n} g^{jkl}\partial_j\partial_k u\, \partial_l v\Big| \le C(1+|x|)^{-1}|\partial v|_x \sum_{k=0}^{n} |\partial\partial_k u|_x. \tag{6.6.7}$$

Proof. We may assume that $|x| > 1$ and that $\sum_0^n |\partial\partial_k u|_x = 1$, $|\partial v|_x = 1$. Write

$$\partial v(x) = V(x) + tA'(x), \quad \partial\partial_k u(x) = U^k(x) + t_k A'(x),$$

with $V(x), U^k(x)$ orthogonal to $A'(x)$. Then we have a bound for

$$|x||V(x)| + |tx| + |tA(x)| + \sum_{k=0}^{n} |x||U^k(x)| + \sum_{k=0}^{n} |t_k x| + \sum_{k=0}^{n} |t_k A(x)|,$$

as in the proof of Lemma 6.6.3. As there we must estimate

$$M = \sum_{j,k,l=0}^{n} g^{jkl} t_k \partial A/\partial x_j t \partial A/\partial x_l.$$

For $\nu = 0, \dots, n$ we have

$$t_k \partial A/\partial x_\nu + U^k_\nu = \partial_\nu \partial_k u = t_\nu \partial A/\partial x_k + U^\nu_k, \quad \text{hence}$$
$$M\partial A/\partial x_\nu = t_\nu t \sum_{j,k,l} g^{jkl}\partial A/\partial x_k \partial A/\partial x_j \partial A/\partial x_l + \sum_{j,k,l} g^{jkl}(U^\nu_k - U^k_\nu)\partial A/\partial x_j t \partial A/\partial x_l.$$

The second sum is bounded and the first sum is $\le C|A(x)||x|$ by (6.6.3). Hence $M|x|$ is bounded and the lemma follows.

The next lemma describes how the vector fields (6.2.8), (6.2.9) act on expressions such as that estimated in Lemma 6.6.4.

Lemma 6.6.5. *Let G be a k linear form on $\mathbf{R}^{1+n}$, and let $k = k_1 + \cdots + k_r$ where k_j are positive integers. Then*

$$G(u_1^{(k_1)}, \dots, u_r^{(k_r)}), \quad u_j \in C^k(\mathbf{R}^{1+n}),$$

is well defined by extension of the k linear form to the tensor products. If Z is a vector field with affine linear coefficients, $Z(x, \partial) = \sum_0^n Z_j(x)\partial_j$, and $u_j \in C^{k+1}(\mathbf{R}^{1+n})$, then

$$ZG(u_1^{(k_1)}, \dots, u_r^{(k_r)}) = G((Zu_1)^{(k_1)}, \dots, u_r^{(k_r)}) + \dots$$
$$+ G(u_1^{(k_1)}, \dots, (Zu_r)^{(k_r)}) + G_1(u_1^{(k_1)}, \dots, u_r^{(k_r)}),$$

where

$$G_1(\xi, \dots, \xi) = \{Z(x, \xi), G(\xi, \dots, \xi)\} = -\sum_0^n \partial Z/\partial x_j \partial G(\xi, \dots, \xi)/\partial \xi_j.$$

Here $\{\, , \}$ denotes the Poisson bracket. If $G(\xi, \dots, \xi) = 0$ when $\xi_0^2 - \cdots - \xi_n^2 = 0$ then this follows for G_1 too if Z is one of the vector fields (6.2.8), (6.2.9) or a constant vector field.

Proof. By Leibniz' rule we must let Z act separately on each of the $u_j^{(k_j)}$, and

$$Z\partial^\alpha = \partial^\alpha Z + Q_\alpha(\partial), \quad \text{where} \quad Q_\alpha(\xi) = -\sum_0^n \partial Z(x, \xi)/\partial x_j \partial \xi^\alpha / \partial \xi_j.$$

The formula for G_1 follows by another application of Leibniz' rule, now for the Poisson bracket. If Z is one of the vector fields (6.2.8), say $Z(x, \xi) = \lambda_j x_j \xi_k - \lambda_k x_k \xi_j$, then

$$\sum_{i=0}^n \partial Z/\partial x_i \partial/\partial \xi_i = \lambda_j \lambda_k (\lambda_k \xi_k \partial/\partial \xi_j - \lambda_j \xi_j \partial/\partial \xi_k),$$

and if Z is the vector field Z_0 in (6.2.9) we obtain the same vector field in the ξ variables. They are all tangent to the light cone, and when Z is a constant vector field we obtain the zero vector field, so $G_1(\xi, \dots, \xi) = 0$ on the light cone if $G(\xi, \dots, \xi) = 0$ there. This proves the last statement.

We shall now discuss the modifications of the energy identity (6.3.3)″ which are required when the coefficients are variable but close to constants. Thus consider an equation of the form

$$\sum_{j,k=0}^n g^{jk}(x)\partial_j \partial_k u = f, \tag{6.6.8}$$

where $n \geq 3$ and $\sum_{j,k=0}^n g^{jk}(x)\partial_j\partial_k = \Box + \sum_{j,k=0}^n \gamma^{jk}(x)\partial_j\partial_k$ with γ^{jk} small. We shall use the same vector field K as in Lemma 6.3.4 apart from an addition of $\partial/\partial t$ to get control of the conventional energy form. Thus we set

(6.6.9)
$$Lu = \sum_0^n L^i \partial_i u + (n-1)x_0 u,$$
$$\hat{L} = (L^0, \dots, L^n) = (1 + x_0^2 + |\vec{x}|^2, 2x_0x_1, \dots, 2x_0x_n),$$

and shall derive a modification of (6.3.3)″,

$$(6.6.10)\quad 2Lu\sum_{j,k=0}^{n} g^{jk}\partial_j\partial_k u=\sum_{j=0}^{n}\partial_j\Big(\sum_{i=0}^{n}T_i^j(u)L^i+(n-1)(2x_0u\sum_{k=0}^{n}g^{jk}\partial_k u-\theta^j u^2)\Big)-R,$$

where $\theta=(1,0,\dots,0)$ and T_i^j is defined by (6.3.4). We shall write down the error term R later on. The energy integrand, that is, the 0 component of the vector field in the divergence on the right, now becomes

$$\begin{aligned}&\sum_i T_i^0(u)L^i+2(n-1)x_0u\sum_k g^{0k}\partial_k u-(n-1)u^2\\ &=2\sum_k g^{0k}\partial_k u\sum_i L^i\partial_i u-\sum_{j,k}g^{jk}\partial_j u\partial_k uL^0+2(n-1)x_0u\sum_k g^{0k}\partial_k u-(n-1)u^2,\end{aligned}$$

so for the energy $E(u)$ defined by integration over $\vec{x}$, we have if $E_0(u)$ denotes the same energy for the unperturbed wave operator,

$$(6.6.11)\quad \begin{aligned}E(u)-E_0(u)=\int\Big(2\sum_{k=0}^{n}\gamma^{0k}\partial_k u\sum_{i=0}^{n}L^i\partial_i u\\ -(1+x_0^2+|\vec{x}|^2)\sum_{j,k=0}^{n}\gamma^{jk}\partial_j u\partial_k u+2(n-1)x_0u\sum_{k=0}^{n}\gamma^{0k}\partial_k u\Big)\,d\vec{x}.\end{aligned}$$

By Lemma 6.3.4

$$(6.6.12)\quad |\langle\hat{L},\xi\rangle|\le C(1+|x|)|\xi|_x.$$

If we assume that

$$(6.6.13)\quad \Big|\sum_{j,k=0}^{n}\gamma^{jk}(x)\xi_j r_k\Big|\le\delta(1+|x|)^{-2}|\xi|_x|r|_x$$

for some small δ, then the integral of the middle sum in (6.6.11) is $O(\delta E_0(u))$, for it follows from Lemma 6.3.5 that

$$(6.6.14)\quad 1/41\le E_0(u)/(\|Z_0u\|^2+\sum_{j<k}\|Z_{jk}u\|^2+\sum_j\|\partial_j u\|^2+\|(n-1)u\|^2)\le 2.$$

When $r_i=\delta_{ik}$ it follows from (6.6.13) that

$$(6.6.13)'\quad \Big|\sum_{j=0}^{n}\gamma^{jk}(x)\xi_j\Big|\le C\delta(1+|x|)^{-1}|\xi|_x,$$

which combined with (6.6.12) proves that also the first and the third sum in (6.6.11) are $O(\delta E_0(u))$. Hence

$$(6.6.15)\quad (1-C\delta)E_0(u)\le E(u)\le(1+C\delta)E_0(u)$$

which shows the equivalence of $E(u)$ and $E_0(u)$ when δ is small.

Next we shall prove that the error term R in (6.6.10) is equal to $\sum_1^6 R_j$ where

$$(6.6.16)\quad \begin{aligned} R_1 &= 2\sum_{j,k,i}\gamma^{jk}\partial_k u(\partial_j L^i)\partial_i u - \sum_i(\partial_i L^i)\sum_{j,k}\gamma^{jk}\partial_j u\partial_k u,\\ R_2 &= 2\sum_i L^i\partial_i u\sum_{j,k}(\partial_j\gamma^{jk})\partial_k u, \quad R_3 = -\sum_{i,j,k}(L^i\partial_i\gamma^{jk})\partial_j u\partial_k u,\\ R_4 &= 2(n-1)u\sum_k\gamma^{0k}\partial_k u, \quad R_5 = 2(n-1)x_0\sum_{j,k}\gamma^{jk}\partial_j u\partial_k u,\\ R_6 &= 2(n-1)x_0 u\sum_{j,k}(\partial_j\gamma^{jk})\partial_k u. \end{aligned}$$

To do so we shall consider successively the terms where ∂_j acts on the various terms and factors in the divergence expression in (6.6.10).

i) The terms where ∂_j acts on a derivative of u in $T_i^j(u)$ occur on the left of (6.6.10).

ii) When ∂_j acts on L^i we obtain a quadratic form in ∂u which is the sum of that in the unperturbed case and R_1.

iii) When ∂_j acts on the components of g in $T_i^j(u)$ we obtain the error terms R_2 and R_3.

iv) If ∂_j acts on $\partial_k u$ in the last sum in (6.6.10) we obtain the remaining part of the left-hand side of (6.6.10).

v) If ∂_j acts on x_0 we obtain the term R_4 in addition to the term $2(n-1)u\partial_0 u$ which occurs in the unperturbed case.

vi) If ∂_j acts on g^{jk} we obtain the term R_6.

vii) If ∂_j acts on u we obtain the term R_5 in addition to the quadratic form in ∂u which occurs in the unperturbed case.

viii) If ∂_j acts on the last term $-(n-1)\theta^j u^2$ we obtain the term $-2(n-1)u\partial_0 u$.

In the unperturbed case we know from (6.3.3)″ that the terms not accounted for in the left-hand side of (6.6.10) or in R cancel out, which proves (6.6.10).

The error terms R_1, R_4 and R_5 can be estimated using (6.6.13). From (6.6.9) we obtain $\sum_0^n \partial_i L^i = 2(n+1)x_0$ and

$$(6.6.17)\quad \sum_{i=0}^n \xi_i\partial_j L^i = \begin{cases} 2\sum_{i=0}^n \xi_i x_i, & \text{when } j=0\\ 2(\xi_0 x_j + \xi_j x_0), & \text{when } j\neq 0.\end{cases}$$

These sums are bounded by $2|\xi|_x$, so it follows from (6.6.13) and (6.6.13)′ that

$$(6.6.18)\quad |R_1|+|R_4|+|R_5| \le C\delta(1+|x|)^{-1}(|\partial u|_x^2 + |u||\partial u|_x).$$

Now we add to (6.6.13) the hypothesis

$$(6.6.19)\quad \Big|\sum_{j,k=0}^n \xi_k\partial_j\gamma^{jk}(x)\Big| \le \delta(1+|x|)^{-2}|\xi|_x.$$

By (6.6.12) the estimate (6.6.18) is then also valid for R_6 and R_2. To obtain such a bound for R_3 we also require

$$(6.6.20)\quad \Big|\sum_{i,j,k=0}^n (L^i\partial_i\gamma^{jk}(x))\xi_j\xi_k\Big| \le \delta(1+|x|)^{-1}|\xi|_x^2.$$

Then we have the estimate (6.6.18) for all R_j, and we have proved

Proposition 6.6.6. *Let $u \in C^2$ be a solution of* (6.6.8) *vanishing for large $|\vec{x}|$ when t is bounded, and assume that* (6.6.13), (6.6.19) *and* (6.6.20) *hold. If δ is sufficiently small it follows that the energy $E(u;x_0)$ at time x_0 obtained by integrating* (6.6.10) *with respect to $\vec{x}$ satisfies*

$$1/50 \le E(u)/(\|Z_0u\|^2 + \sum \|Z_{jk}u\|^2 + \sum \|\partial_j u\|^2 + \|(n-1)u\|^2) \le 3. \tag{6.6.21}$$

We have

$$\begin{aligned}|\partial E(u;x_0)/\partial x_0| \le C\Big(&\delta(1+x_0)^{-1}E(u;x_0)\\ &+ \big(\int (1+x_0+|\vec{x}|)^2|f(x_0,\vec{x})|^2\,d\vec{x}\big)^{\frac12}E(u;x_0)^{\frac12}\Big).\end{aligned} \tag{6.6.22}$$

Here we have used (6.6.12) when estimating the left-hand side of the energy identity (6.6.10) and of course used that the integrals of the terms on the right with $j \ne 0$ are equal to 0.

After introducing $E(u;x_0)^{\frac12}$ as a new unknown in (6.6.22) and multiplication of both sides by $(1+x_0)^{-C\delta}$ we can integrate and obtain

$$\begin{aligned}E(u;x_0)^{\frac12} \le (1+x_0)^{C\delta}\big(&E(u;0)^{\frac12}\\ &+ C\int_0^{x_0}(1+t)^{-C\delta}dt\big(\int (1+t+|\vec{x}|)^2|f(t,\vec{x})|^2\,d\vec{x}\big)^{\frac12}\big).\end{aligned} \tag{6.6.23}$$

We shall use (6.6.22) instead of (6.3.6) when we study the equation (6.6.1). The first order terms in g^{jk} will then have the special properties in Definition 6.6.1. To apply Proposition 6.6.6 we need the following

Lemma 6.6.7. *Suppose that*

$$\gamma^{jk}(x) = \sum_{l=0}^{n} g^{jkl}\partial_l w(x) + \varrho^{jk}(x), \tag{6.6.24}$$

where g^{jkl} are constants satisfying the null condition (6.6.3) *and*

$$(1+|x|)\sum_{|I|\le 2}|Z^I w(x)| \le \delta, \quad (1+|x|)^2\sum_{|I|\le 1}|Z^I\varrho^{jk}(x)| \le \delta, \tag{6.6.25}$$

where Z^I is any product of $|I|$ vector fields of the form (6.2.8), (6.2.9) *or $\partial/\partial x_j$. Then* (6.6.13), (6.6.19) *and* (6.6.20) *are valid with δ replaced by a constant times δ.*

Proof. This is obvious when $\gamma^{jk} = \varrho^{jk}$, if we recall (6.6.12) when verifying (6.6.20). We may therefore assume that $\varrho^{jk} = 0$. To prove (6.6.13) we use that by Lemma 6.6.3

$$\Big|\sum_{j,k,l=0}^{n} g^{jkl}\partial_l w\xi_j r_k\Big| \le C(1+|x|)^{-1}|\partial w|_x|\xi|_x|r|_x,$$

and that $|\partial w|_x \le C\delta(1+|x|)^{-1}$ by (6.6.25). By Lemma 6.6.4

$$\Big|\sum_{j,k=0}^{n}\xi_k\partial_j\gamma^{jk}(x)\Big| = \Big|\sum_{j,k,l=0}^{n} g^{jkl}\partial^2 w/\partial x_l\partial x_j\xi_k\Big| \le C(1+|x|)^{-1}|\xi|_x\sum_{l=0}^{n}|\partial\partial_l w|_x,$$

which proves (6.6.19). To prove (6.6.20) we write, now with $\hat{L} = \sum_0^n L^i \partial/\partial x_i$,

$$\sum_{j,k=0}^{n} (\hat{L}\gamma^{jk})\xi_j\xi_k = \sum_{j,k,l=0}^{n} g^{jkl}\xi_j\xi_k\partial_l(\hat{L}w) + \sum_{j,k,l=0}^{n} g^{jkl}\xi_j\xi_k[\hat{L},\partial_l]w.$$

By Lemma 6.6.3 the first sum can be estimated by

$$C(1+|x|)^{-1}|\xi|_x^2|\partial\hat{L}w|_x,$$

and $|\partial\hat{L}w|_x \le C\delta$ by (6.6.12) and (6.6.25). Since $[\partial_0, \hat{L}] = 2Z_0$ and $[\partial_l, \hat{L}] = 2Z_{0l}$ for $l \neq 0$, we have by (6.6.25)

$$|[\hat{L},\partial_l]w| \le C\delta(1+|x|)^{-1},$$

which completes the proof of (6.6.20) also.

We shall need an analogue of Proposition 6.5.1 involving L^1 norms, which also follows from Sobolev's inequality. For the sake of simplicity we assume from now on that $n = 3$. (See also Hörmander [3] for general $n \ge 3$.)

Lemma 6.6.8. *If $u \in C^4$ and $\Box u = F$ in $[0,t] \times \mathbf{R}^3$, and the Cauchy data of u are 0 when $t = 0$, then*

$$(6.6.26) \qquad (1+t+|\vec{x}|)|u(t,\vec{x})| \le C\iint_{0<s<t} \sum_{|I|\le 2} |Z^I F(s,\vec{y})|\, ds\, d\vec{y}/(1+s+|\vec{y}|),$$

where Z^I is any product of $|I|$ vector fields of the form (6.2.8), (6.2.9) *or* $\partial/\partial x_j$.

Proof. We shall first prove a homogeneous version of (6.6.26): if F vanishes in a neighborhood of the origin then

$$(6.6.26)' \qquad (t+|\vec{x}|)|u(t,\vec{x})| \le C\iint_{0<s<t} {\sum_{|I|\le 2}}' |Z^I F(s,\vec{y})|\, ds\, d\vec{y}/(s+|\vec{y}|),$$

with summation only over products of vector fields of the form (6.2.8), (6.2.9) which preserve homogeneity. A change of scales shows that it suffices to prove (6.6.26)′ when $t = 1$.

The solution u of the wave equation is given by the retarded potential

$$u(t,\vec{x}) = (4\pi)^{-1}\int_{|\vec{y}|<t} F(t-|\vec{y}|, \vec{x}-\vec{y})\, d\vec{y}/|\vec{y}|.$$

The difficulties in the proof come from the fact that the integration only takes place over a hypersurface and that the denominator $|\vec{y}|$ vanishes when $\vec{y} = 0$.

i) Let us first assume that $\operatorname{supp} F \subset \{(s,\vec{y}); |\vec{y}| < \frac{1}{2}s\}$. In this set the vector fields Z used in (6.6.26)′ span all vector fields, and we conclude that

$$(6.6.27) \qquad \iint_{0<s<1} \sum_{|\alpha|\le 2} s^{|\alpha|}|\partial^\alpha F(x,\vec{y})|\, dx\, d\vec{y}/s$$

$$\le C\iint_{0<s<1} {\sum_{|I|\le 2}}' |Z^I F(s,\vec{y})|\, ds\, d\vec{y}/(s+|\vec{y}|).$$

We may assume in the proof of (6.6.26)′ with $t=1$ that $|\vec{x}|\le 1$ since $u(1,\vec{x})=0$ otherwise. When $|\vec{y}|>\frac12$ we use the estimate

$$|F(1-|\vec{y}|,\vec{x}-\vec{y})|\le\int_0^1(|F(s,\vec{x}-\vec{y})|+|F_s'(s,\vec{x}-\vec{y})|)\,ds,$$

and $1/|\vec{y}|<2$ then. When $|\vec{y}|\le\frac12$ we use instead the similar estimate

$$|F(1-|\vec{y}|,\vec{x}-\vec{y})|\le\int_{\frac12}^1(2|F(s,\vec{x}-\vec{y})|+|F_s'(s,\vec{x}-\vec{y})|)\,ds,$$

which avoids small values of s, and then we apply the estimate

$$\int|g(\vec{y})|\,d\vec{y}/|\vec{y}|\le\tfrac12\int|g'(\vec{y})|\,d\vec{y},\quad g\in C_0^1(\mathbf{R}^3),$$

which follows by introducing polar coordinates and observing that integration by parts gives

$$\int_0^\infty|G(r)|r\,dr\le\tfrac12\int_0^\infty|G'(r)|r^2\,dr,\quad G\in C_0^1(\mathbf{R}).$$

Summing up, we have proved that

$$4\pi|u(1,\vec{x})|\le 2\iint_{0<s<1}(|F(s,\vec{y})|+|F'(s,\vec{y})|+s|F''_{s\vec{y}}(s,\vec{y})|)\,ds\,d\vec{y},$$

which gives (6.6.26)′ when combined with (6.6.27).

ii) Assume now that $\operatorname{supp}F\subset\{(s,\vec{y});|\vec{y}|>\frac13 s\}$. By Sobolev's lemma (see (6.4.13)‴)

$$M(t,r)=\sup_{|\omega|=1}|F(t,r\omega)|\le C\sum_{|I|\le 2}{}''\int_{|\omega|=1}|(Z^IF)(t,r\omega)|\,d\omega,$$

where the summation only contains products of the vector fields (6.2.8) corresponding to Euclidean rotations. Hence

$$\iint_{0<s<1,\varrho>0}M(s,\varrho)\varrho\,ds\,d\varrho\le C\iint_{0<s<1}\sum_{|I|\le2}{}''|Z^IF(s,\vec{y})|\,ds\,d\vec{y}/(s+|\vec{y}|).\tag{6.6.28}$$

Replacing f by M will increase the retarded potential, so $|u|\le U$ where $\Box U=M$ and U has Cauchy data zero. It is clear that U is rotationally symmetric in the space variables, and expressing $\Box$ in polar coordinates we have

$$(\partial_t^2-\partial_r^2)rU(t,r)=rM(t,r),$$

which implies that

$$|\vec{x}||u(1,\vec{x})|\le rU(1,r)\le\tfrac12\iint_{0<s<1,\varrho>0}M(s,\varrho)\varrho\,ds\,d\varrho.$$

Combined with (6.6.28) this proves (6.6.26)′ when $|\vec{x}|\ge\frac14$.

We shall now give a different proof when $|\vec{x}| < \frac{1}{4}$. If $(1 - |\vec{y}|, \vec{x} - \vec{y}) \in \operatorname{supp} F$ we have $3|\vec{x} - \vec{y}| > 1 - |\vec{y}|$, hence $4|\vec{y}| > 1 - 3|\vec{x}| > \frac{1}{4}$ and

$$(6.6.29) \qquad 4\pi|u(1,\vec{x})| \le 16 \int_{1/16<|\vec{y}|<1} |F(1-|\vec{y}|, \vec{x}-\vec{y})|\,d\vec{y}.$$

We shall now use that we control the radial derivative Z_0F of F. To do so we write $\varphi(\tau,\vec{y}) = \tau(1 - |\vec{y}|, \vec{x} - \vec{y})$ where $1/16 \le |\vec{y}| \le 1$ and $1 \le \tau \le 16/15$. The Jacobian $\tau^n(1 - \langle \vec{x}, \vec{y}\rangle/|\vec{y}|)$ is bounded below by $3/4$, and we have

$$\int |F(\varphi(1,\vec{y}))|\,d\vec{y} \le \int_{1/16<|\vec{y}|<1, 1<\tau<16/15} (15|F(\varphi(\tau,\vec{y}))| + |\partial F(\varphi(\tau,\vec{y}))/\partial\tau|)\,d\tau\,d\vec{y}.$$

Since $\partial F(\varphi(\tau,\vec{y}))/\partial\tau = (Z_0F)(\varphi(\tau,\vec{y}))/\tau$, we obtain by changing to the integration variables $\varphi(\tau,\vec{y})$ and combining the estimate with (6.6.29) that

$$|u(1,\vec{x})| \le C \iint_{0<s<1, |\vec{y}|<2} (|F(s,\vec{y})| + |Z_0F(s,y)|)\,ds\,d\vec{y}, \quad |\vec{x}| < \tfrac{1}{4},$$

which completes the proof of (6.6.26)′ in this case.

Choose $\psi \in C_0^\infty(\mathbf{R}^3)$ so that $|\vec{y}| < \frac{1}{2}$ in the support of ψ and $|\vec{y}| > 1/3$ in the support of $1 - \psi$. Then $\psi(\vec{y}/s)F(s,\vec{y})$ and $(1 - \psi(\vec{y}/s))F(s,\vec{y})$ satisfy the hypothesis in i) and ii) respectively. Since the operators Z in (6.6.26)′ applied to $\psi(\vec{y}/s)$ give a function which is homogeneous of degree 0, hence bounded, the estimate (6.6.26)′ follows from the two cases proved above.

If $\operatorname{supp} F \subset \{(s,\vec{y}); s + |\vec{y}| \ge 1\}$, then the same inclusion is true for $\operatorname{supp} u$, so (6.6.26) is valid in that case. On the other hand, if $s + |\vec{y}| \le 2$ in $\operatorname{supp} F(s,\vec{y})$ we obtain (6.6.26) by applying the case already proved to a translation such as $F(s, y_1 + 3, y_2, y_3)$. (The translation introduces the constant vector fields.) Combining the two cases by a partition of unity yields (6.6.26) in full generality.

Proof of Theorem 6.6.2. Let k be an integer ≥ 5, and assume that we already have a C^∞ solution of (6.6.1), (6.6.2) for $0 \le x_0 \le T$ such that for such x_0 and small ε

$$(6.6.30) \qquad (1 + |x|) \sum_{|I|\le k} |Z^I u(x)| \le C_0\varepsilon,$$

$$(6.6.31) \qquad \sum_{|I|\le k+4} \|Z^I u(x_0,\cdot)\| \le C_1(1 + x_0)^{C_2\varepsilon} \sum_{|I|\le k+4} \|Z^I u(0,\cdot)\|.$$

Let C_0 be so large that (6.6.30) holds with C_0 replaced by $C_0/3$ if u is replaced by the solution εu^0 of the wave equation with Cauchy data (6.6.2). We shall then prove for ε smaller than some number depending on C_1 and C_2 that

i) (6.6.30) is valid with C_0 replaced by $C_0/2$;

ii) (6.6.31) is a consequence of (6.6.30) for suitable C_j.

By the local existence theorem (Theorem 6.4.11) it will follow that a solution exists for all $x_0 \ge 0$ if ε is small enough.

i) Since the Cauchy data of $Z^I u - Z^I \varepsilon u^0$ are $O(\varepsilon^2)$, it suffices by Lemma 6.6.8 to prove that for small ε

$$\sum_{|I|\le k} \sum_{|J|\le 2} \iint_{0<t<T} |Z^J \Box Z^I u(t,\vec{x})|\,dt\,d\vec{x}/(1+t) \le C\varepsilon^2.$$

We can write $Z^J \Box Z^I u$ as a sum of terms of the form $Z^K \Box u$ with $|K| \le |J| + |I| \le k+2$, so it suffices to prove that

$$\sum_{|K|\le k+2} \iint_{0<t<T} |Z^K \Box u(t,\vec{x})| \, dt d\vec{x}/(1+t) \le C\varepsilon^2. \tag{6.6.32}$$

To do so we write (6.6.1) in the form

$$\Box u = -\sum_{j,k=0}^{n} \gamma^{jk}(u,u')\partial_j\partial_k u + f(u,u'). \tag{6.6.33}$$

When we apply Z^K to (6.6.33) we obtain a number of terms containing products of derivatives of u of order $\le k+4$, and two factors can never be of order $> k$ since each such factor requires $k-1$ factors Z acting on u'' or k factors Z acting on (u,u'), thus altogether $k-1+k > k+2$ factors Z. Apart from the quadratic terms we can factor so that we must have three factors $Z^I u$. We estimate the two factors with highest $|I|$ using the L^2 estimate (6.6.31) and the others by means of (6.6.30). This shows that the L^1 norm can be estimated by $C\varepsilon^3(1+t)^{2C_2\varepsilon-1}$. For the quadratic terms

$$Z^K \sum_{j,k,l} g^{jkl}\partial_j\partial_k u\,\partial_l u, \quad Z^K \sum_{j,k} f^{jk}\partial_j u \partial_k u$$

we first use Lemma 6.6.5 to obtain a sum of terms of this form with the factors Z next to u. We can then apply Lemma 6.6.3 or Lemma 6.6.4 together with the L^2 estimate (6.6.31). It follows that

$$\sum_{|K|\le k+2} \int |Z^K \Box u(t,\vec{x})| \, d\vec{x} \le C\varepsilon^2(1+t)^{2C_2\varepsilon-1},$$

and this implies (6.6.32) when $2C_2\varepsilon < 1$.

ii) To prove (6.6.31), assuming (6.6.30) known, we shall apply the energy estimates in Proposition 6.6.6 to the equations obtained when (6.6.1) is multiplied by Z^I for $|I| \le k+3$,

$$\sum_{j,k=0}^{n} g^{jk}(u,u') Z^I u = \sum_{j=0}^{4} f_I^j, \tag{6.6.34}$$

where

$f_I^0 = [\Box, Z^I]u$;

f_I^1 is a sum of terms of the form $\widehat{G}((Z^J u)', (Z^K u)'')$ with $|J| + |K| \le |I|$, $|K| < |I|$, and a trilinear form $\widehat{G}$ satisfying the null condition (cf. Lemma 6.6.5);

$f_I^2 = \sum c_{IJK}(Z^J \varrho^{jk}(u,u'))Z^K \partial_j\partial_k u$ with the same conditions on J and K, and ϱ^{jk} vanishing of second order at 0;

f_I^3 is a sum of terms of the form $\widehat{F}((Z^K u)', (Z^J u)')$ with $|J| + |K| \le |I|$ and a bilinear form $\widehat{H}$ satisfying the null condition (cf. Lemma 6.6.5);

$f_I^4 = Z^I R(u,u')$ where R vanishes of third order at 0.

We shall consider $g^{jk}(u,u')$ as a function of x when we apply Proposition 6.6.6. It follows from (6.6.30) and Lemma 6.6.7 that the hypotheses of Proposition 6.6 are then fulfilled with δ equal to a constant times ε. Let

$$E_k(u;x_0) = \sum_{|I|\le k+3} E(Z^I u; x_0)$$

with E defined as in Proposition 6.6.6; by (6.6.21) this is equivalent to the square of left-hand side of (6.6.31). We shall prove that

$$\int(1+x_0+|\vec{x}|)^4|f_I^j(x_0,\vec{x})|^2d\vec{x}\le C\varepsilon^2E_k(u;x_0),\quad |I|\le k+3,\ 0\le j\le 4. \tag{6.6.35}$$

To estimate f_I^j we note that no term can contain more than one factor Z^Ju with $|J|>k$, for such a factor is only obtained when at least $k-1$ operators Z act on $\partial_j\partial_k u$ or at least k of them act on (u,u'), and this would add up to at least $2k-1>k+3$ operators. (This is where we need that $k\ge 5$.) We factor terms vanishing of second or third order using Taylor's formula. In each term we can estimate all factors except one using (6.6.30). For the third order terms f_I^2 and f_I^4 this gives a factor $\le C\varepsilon^2(1+x_0+|\vec{x}|)^{-2}$ in addition to a factor with norm square $O(E_k(u;x_0))$. For the second order terms f_I^1 and f_I^3 we obtain in addition to the factor $\varepsilon/(1+|x|)$ from (6.6.30) a factor $(1+|x|)^{-1}$ by using Lemmas 6.6.3 and 6.6.4 as in part i) of the proof. To estimate f_I^0, finally, we write

$$[\Box,Z^I]u=\sum_{|J|<|I|}c_{IJ}Z^J\Box u$$

and replace $\Box u$ by the expression in (6.6.33). The terms then obtained are similar to those already discussed in f_I^j with $j\ne 0$ but they contain one derivative less, which proves (6.6.35).

From (6.6.35) and Proposition 6.6.6 it follows that

$$\partial E_k(u;x_0)/\partial x_0\le C\varepsilon(1+x_0)^{-1}E_k(u;x_0),$$

hence that

$$E_k(u;x_0)\le(1+x_0)^{C\varepsilon}E_k(u;0).$$

This proves that (6.6.31) follows from (6.6.30) with constants independent of x_0 which completes the proof.

6.7. Global existence theorems by the conformal method. Suppose that in some open subset of Minkowski space we have a C^2 solution of a differential equation

$$\Box u+f(u,u',u'')=0 \tag{6.7.1}$$

where f vanishes of second order at the origin. This means that $u=0$ is a solution where the linearization is the wave operator in Minkowski space. In Section A.4 of the appendix we have defined a conformal isomorphism Ψ between a bounded part of the Einstein universe $\mathbf{R}\times S^n$ and the Minkowski space $\mathbf{R}^{1+n}$ with the Lorentz metric. (See (A.4.2) and Theorem A.4.1.) On the inverse image in the Einstein universe of the domain of u we define a function $\tilde{u}$ by

$$\tilde{u}(T,X)=\Omega(T,X)^{\frac{1-n}{2}}u(\Psi(T,X)),$$

and obtain using (A.4.3) and (A.3.7), with n replaced by $n+1$ and $e^\varphi=\cos T+X_0=\Omega$,

$$(\tilde{\Box}-\tilde{S}(n-1)/4n)\tilde{u}=\Omega^{-\frac{n+3}{2}}(\Box u)\circ\Psi,$$

where $\tilde{\Box}$ is the d'Alembertian on $\mathbf{R}\times S^n$ and $\tilde{S}$ is the scalar curvature there, that is, $\tilde{S}=-n(n-1)$ by (A.2.10). Hence (6.7.1) is equivalent to

$$(\tilde{\Box}+(n-1)^2/4)\tilde{u}+\Omega^{-\frac{n+3}{2}}f(u,u',u'')\circ\Psi=0. \tag{6.7.2}$$

Here we should interpret $u \circ \Psi$ as $\Omega^{\frac{n-1}{2}}\tilde{u}$ and express the differentiations using (A.4.6), (A.4.6)$'$. In view of (A.4.7) this means that every component of u' (resp. u'') is the product of $\Omega^{\frac{n-1}{2}}$ and a sum with analytic coefficients of derivatives of $\tilde{u}$ of order ≤ 1 (resp. ≤ 2). By Taylor's formula f can be written as a quadratic form in (u, u', u'') with coefficients which are C^∞ functions of (u, u', u''). Any product $\partial^\alpha u \partial^\beta u$, $0 \leq |\alpha| \leq 2$, $0 \leq |\beta| \leq 2$, gives the product of Ω^{n-1} and a quadratic form with analytic coefficients in the derivatives of $\tilde{u}$. If $n - 1 - (n+3)/2 \geq 0$, that is, $n \geq 5$, and n is *odd*, it follows that (6.7.1) is equivalent to

$$(6.7.3) \qquad (\tilde{\Box} + (n-1)^2/4)\tilde{u} + \tilde{f}(T, X, \tilde{u}, \tilde{u}', \tilde{u}'') = 0,$$

where $\tilde{f}$ is an analytic function of T, X and $U = (\tilde{u}, \tilde{u}', \tilde{u}'')$ for small U, vanishing of second order when $U = 0$.

For *even* $n \geq 4$ we encounter half integer powers of Ω, and it is easy to see that we get a C^∞ (analytic) function $\tilde{f}$ precisely when f is odd. This is of course far less useful than the case of odd n, so we shall now look in more detail at what happens when $n = 3$. If f vanishes of third order at $(0,0,0)$, then $\tilde{f}$ has no singularity for $\Omega = 0$ if $3(n-1)/2 \geq (n+3)/2$, that is, $n \geq 3$. It is therefore sufficient to examine for which quadratic forms f in (u, u', u''),

$$f = \sum_{|\alpha|,|\beta|\leq 2} f_{\alpha\beta}\partial^\alpha_{t,x} u \, \partial^\beta_{t,x} u$$

that the quadratic form $\tilde{f}$ in $(\tilde{u}, \tilde{u}', \tilde{u}'')$ defined by

$$\tilde{f} = \Omega^{-3} \sum_{|\alpha|,|\beta|\leq 2} f_{\alpha\beta}\partial^\alpha_{t,x}(\Omega\tilde{u}) \, \partial^\beta_{t,x}(\Omega\tilde{u})$$

has regular coefficients also when $\Omega = 0$. Expressing the derivatives by (A.4.6), (A.4.6)$'$ and using (A.4.7) we see at once that $G(\tilde{u}, \tilde{u}', \tilde{u}'') = \Omega\tilde{f}$ has analytic coefficients which as functions of T are trigonometric polynomials of degree ≤ 4. Now an analytic function c on $\mathbf{R} \times S^3$ which is a trigonometrical polynomial in T can be written $c = \Omega d$ for some d of the same class if (and only if) $c = 0$ on $\Sigma = \{(T, X); \cos T + X_0 = 0, 0 < T < \pi\}$. In fact, a trigonometrical polynomial is the sum of a polynomial in $\cos T$ and such a polynomial multiplied by $\sin T$, so polynomial division gives

$$c = \Omega d + c_1 \sin T + c_2$$

where c_1 and c_2 are analytic functions on S^3. If $c = 0$ on Σ then

$$c_1(X)^2(\vec{X}, \vec{X}) = c_2(X)^2, X = (X_0, \vec{X}) \in S^n.$$

This remains true for small complex values of $\vec{X}$ when $X_0 = \sqrt{1 - (\vec{X}, \vec{X})}$ which gives a contradiction when $\vec{X}$ is close to a zero $\neq 0$ of $(\vec{X}, \vec{X})$ unless c_1 and c_2 vanish identically on S^3, for $(\vec{X}, \vec{X})$ vanishes simply on a complex hypersurface whereas $c_1(X)^2$ and $c_2(X)^2$ must have zeros of even order or vanish identically.

Thus it is sufficient to determine the conditions for all coefficients of G to vanish at Σ. It follows from (A.4.7) that the restriction of $G(\tilde{u})$ to Σ depends only on the restriction of $\tilde{u}$ to Σ. In particular, we can check this using functions $\tilde{u}$ satisfying the homogeneous conformal d'Alembertian and having Cauchy data vanishing near the pole at infinity, that is, functions $\tilde{u}$ corresponding to solutions u of the wave equation in $\mathbf{R}^4$ with Cauchy data in $C_0^\infty(\mathbf{R}^3)$. It follows from (6.2.15) that apart from a change of variables and a fixed factor the restriction of $\tilde{u}$ to Σ is then the Friedlander radiation field F. Now the radiation field $F(\omega, \varrho)$ can have an arbitrary Taylor expansion of order 2 at a given point, for if $F(\omega, \varrho)$ is a radiation field, then so is $F(\omega, \varrho + \varrho_0)$ for constant ϱ_0, $\partial F(\omega, \varrho)/\partial \varrho$ and $\omega_j \partial F(\omega, \varrho)/\partial \varrho$ for any j. Starting from a radial radiation field this allows us to find another with given Taylor expansion at a chosen point. To check whether $G(\tilde{u}, \tilde{u}', \tilde{u}'')$ vanishes on Σ it is thus sufficient to use such functions. That G vanishes on Σ means that $f(u, u', u'') = o(\Omega^2) = o(t^{-2})$ if $x = (t + \varrho)\omega$ and $t \to \infty$. By Theorem 6.2.1 with $n = 3$

$$\partial_{t,x}^{\alpha} u(t, x) = t^{-1} \hat{\omega}^{\alpha} F_{|\alpha|}(\omega, \varrho) + O(t^{-2})$$

where $\hat{\omega} = \varrho' = (-1, \omega)$, and $F_j(\omega, \varrho) = \partial^j F(\omega, \varrho)/\partial \varrho^j$ where F is the Friedlander radiation field of u. Thus we arrive at the condition

$$\sum_{|\alpha|, |\beta| \le 2} f_{\alpha\beta} \hat{\omega}^{\alpha+\beta} F_{|\alpha|}(\omega, \varrho) F_{|\beta|}(\omega, \varrho) = 0. \tag{6.7.4}$$

Since $F_{|\alpha|}(\omega, \varrho)$ can be given arbitrary values at a given point and the rays generated by the vectors $\hat{\omega}$ cover the boundary of the light cone, this means that

$$\sum_{|\alpha|=j, |\beta|=k} f_{\alpha\beta} \xi^{\alpha+\beta} = 0, \quad \text{if } \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0; \quad 0 \le j \le k \le 2. \tag{6.7.4)'$$

(We have assumed here that $f_{\alpha\beta} = f_{\beta\alpha}$ which is no restriction. When $j = 0$ this means that u can only occur in a term $c\Box u$, and u is then easily removed from the equation (6.7.1) if we divide by $(1 + cu)$ which we can do for small u.) We sum up the condition encountered in the following definition:

Definition 6.7.1. When $n = 3$ we say that $f(u, u', u'')$ satisfies the *null condition* if f vanishes of second order at $(0, 0, 0)$ and the quadratic part f_2 satisfies the condition

$$f_2(a_0, a_1\xi, a_2\xi \otimes \xi) = 0, \quad \text{if } a_j \in \mathbf{R},\ \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0. \tag{6.7.5}$$

Here $\xi \otimes \xi = \frac{1}{2}\partial_x^2(\langle x, \xi\rangle^2)$. The condition (6.7.5) is of course just another way of writing (6.7.4), (6.7.4)'. It is an extension of the null condition in Definition 6.6.1, due to Christodoulou [1] and Klainerman [4] who only consider quasilinear equations and required u to be absent in the quadratic terms. As already pointed out, the latter extension is quite trivial though. The following result was proved in Christodoulou [1] in the quasilinear case using the same method as here; an alternative proof using L^2 estimates in Minkowski space due to Klainerman [4] was presented in Section 6.6.

Theorem 6.7.2. *The differential equation* (6.7.1) *in* $\mathbf{R}^{1+n}$ *has a global* C^∞ *solution for arbitrary small Cauchydata in* $\mathcal{S}(\mathbf{R}^n)$ *if* $f \in C^\infty$ *vanishes of second order at* $(0, 0, 0)$ *and* n *is odd and* > 3 *or* $n = 3$ *and* f *satisfies the null condition. For the solution we have*

$$|u(t, x)| \le C\big((1 + (|x| - t)^2)(1 + (|x| + t)^2)\big)^{\frac{1-n}{2}}$$

Proof. The Cauchy problem is equivalent to the Cauchy problem for the equation (6.7.2) where the non-linar term is analytic, depending on (T, X) also, and the Cauchy data are small in $C^\infty(S^n)$. The existence of a solution for $0 \le T \le \pi$ follows from Theorem 6.4.11 and Remarks 3 and 4 after its proof.

Theorem 6.4.11 yields a much more precise existence theorem. Let the Cauchy data be

$$u = u_0, \ \partial_t u = u_1 \quad \text{when } t = 0. \tag{6.7.6}$$

For the equation (6.7.3) we then have the Cauchy data

$$(6.7.6)' \qquad \tilde{u} = \tilde{u}_0,\ \partial_T \tilde{u} = \tilde{u}_1 \quad \text{when } T = 0,$$

where

$$\tilde{u}_0(X) = \Omega(0,X)^{\frac{1-n}{2}} u_0(\Psi(0,X)),\ \tilde{u}_1(X) = \tfrac{1}{2}\Omega(0,X)^{\frac{-1-n}{2}} u_1(\Psi(0,X)),\ X \in S^n.$$

Recall that derivatives in S^n for $T=0$ can be expressed as linear combinations of products of the operators

$$Z_{jk} = x_j\partial/\partial x_k - x_k\partial/\partial x_j;\ j,k = 1,\dots,n;$$
$$L_k = \tfrac{1}{2}(1-|x|^2)\partial/\partial x_k + x_k\langle x, \partial/\partial x\rangle.$$

(The operators L_k occur in the right-hand side of (A.4.10).) We have

$$\Omega(0,X)^{\frac{1-n}{2}} = 2^{\frac{1-n}{2}}(1+|x|^2)^{\frac{n-1}{2}},$$

which is annihilated by all Z_{jk} whereas application of L_k to this factor is equivalent to multiplication by $(n-1)x_k/2$ since

$$L_k(1+|x|^2) = (1-|x|^2)x_k + x_k 2|x|^2 = (1+|x|^2)x_k.$$

A product of κ operators Z_{jk} and L_i is of the form

$$\sum_{1\le|\alpha|\le\kappa} a_\alpha(x)\partial^\alpha,$$

where a_α is a polynomial of degree $\le \kappa + |\alpha|$. This follows by induction over κ. If we now recall that the spherical surface measure is $2^n(1+|x|^2)^{-n}\,dx$ by (A.4.11), which almost compensates for the factor $\Omega(0,X)^{1-n}$, we find that if $u_0 \in H^{\text{loc}}_{(s+1)}(\mathbf{R}^n)$ and $u_1 \in H^{\text{loc}}_{(s)}(\mathbf{R}^n)$ where s is a positive integer, then

$$(6.7.7) \qquad \|\tilde{u}_0\|^2_{(s+1)} \le C \sum_{|\alpha|\le s+1} \int |\partial^\alpha u_0(x)|^2(1+|x|^2)^{s+|\alpha|}\,dx,$$

$$(6.7.8) \qquad \|\tilde{u}_1\|^2_{(s)} \le C \sum_{|\alpha|\le s} \int |\partial^\alpha u_1(x)|^2(1+|x|^2)^{s+|\alpha|+1}\,dx.$$

In the quasilinear case it follows from Theorem 6.4.11 and Remark 3 after its proof that (6.7.1) has a global solution satisfying (6.7.6) provided that the right-hand sides of (6.7.7) and (6.7.8) are well defined and small enough when $s = (n+3)/2$. This is a result of Christodoulou [1]. For the fully non-linear case we obtain the same result with $s = (n+5)/2$ by Remark 4 there. Note that the finiteness of these norms requires that

$$|u_j(x)| \le C(1+|x|)^{-s-j-n/2}, \quad j = 0,1,\ x \in \mathbf{R}^n.$$