Existence and universality of the blow-up profile for the semilinear wave equation in one space dimension

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Abstract

In this paper, we consider the semilinear wave equation with a power nonlinearity in one space dimension. We exhibit a universal one-parameter family of functions which stand for the blow-up profile in self-similar variables at a non-characteristic point, for general initial data. The proof is done in self-similar variables. We first characterize all the solutions of the associated stationary problem, as a one parameter family. Then, we use energy arguments coupled with dispersive estimates to show that the solution approaches this family in the energy norm, in the non-characteristic case, and to a finite decoupled sum of such a solution in the characteristic case. Finally, in the case where this sum is reduced to one element, which is the case for non-characteristic points, we use modulation theory coupled with a nonlinear argument to show the exponential convergence (in the self-similar time variable) of the various parameters and conclude the proof. This step provides us with a result of independent interest: the trapping of the solution in self-similar variables near the set of stationary solutions, valid also for non-characteristic points. The proof of these results is based on a new analysis in the self-similar variable.

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1. Introduction

1.1. The problem and known results

We consider the following one-dimensional semilinear wave equation:

$$\begin{align*}
\partial_{tt}u &= \partial_{xx}u + |u|^{p-1}u, \\
u(0) &= u_0 \text{ and } u_t(0) = u_1,
\end{align*}$$

(1)

where \( u(t) : x \in \mathbb{R} \rightarrow u(x,t) \in \mathbb{R} \), \( u_0 \in H^1_{\text{loc},u} \) and \( u_1 \in L^2_{\text{loc},u} \) with

$$\|v\|_{L^2_{\text{loc},u}}^2 = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 \, dx$$

and

$$\|v\|_{H^1_{\text{loc},u}}^2 = \|v\|_{L^2_{\text{loc},u}}^2 + \|\nabla v\|_{L^2_{\text{loc},u}}^2.$$ 

The Cauchy problem for Eq. (1) in the space \( H^1_{\text{loc},u} \times L^2_{\text{loc},u} \) follows from the finite speed of propagation and the wellposedness in \( H^1 \times L^2 \). See for instance Ginibre et al. [7], Ginibre and Velo [8], Lindblad and Sogge [12] (for the local in time wellposedness in \( H^1 \times L^2 \)). The existence of blow-up solutions for Eq. (1) is a consequence of the finite speed of propagation and ODE techniques (see for example Levine [11] and Antonini and Merle [4]). More blow-up results can be found in Caffarelli and Friedman [5], Alinhac [1,2], Kichenassamy and Littman [9,10] and Shatah and Struwe [21]). Note that an important part of the literature on blow-up in the wave framework is devoted to quasilinear wave equations (where the nonlinearity occurs in the diffusion term). Such equations may develop “geometric” blow-up (see Alinhac [1–3]).

Most of the previous literature considered blow-up for the wave equation from the point of view of prediction. Indeed, most of the papers gave sufficient conditions to have blow-up or constructed special solutions with a prescribed behavior (see [9,10] for example). As we did in our earlier work [17–19], we adopt in this paper a different point of view and aim at describing the blow-up behavior for any blow-up solution. More precisely, this paper is dedicated to the blow-up profile in self-similar variables.

If \( u \) is a blow-up solution of (1), we define (see for example Alinhac [1]) a continuous curve \( \Gamma \) as the graph of a function \( x \rightarrow T(x) \) such that \( u \) cannot be extended beyond the set

$$D_u = \{(x,t) \mid t < T(x)\}.$$ 

(2)

The set \( D_u \) is called the maximal influence domain of \( u \). From the finite speed of propagation, \( T \) is a 1-Lipschitz function. Let \( \bar{T} \) be the infimum of \( T(x) \) for all \( x \in \mathbb{R} \). The time \( \bar{T} \) and the surface \( \Gamma \) are called (respectively) the blow-up time and the blow-up surface of \( u \).

Let us first introduce the following non-degeneracy condition for \( \Gamma \). If we introduce for all \( x \in \mathbb{R}, t \leq T(x) \) and \( \delta > 0 \), the cone

$$\mathcal{C}_{x,t,\delta} = \{(\xi,\tau) \neq (x,t) \mid 0 \leq \tau \leq t - \delta|x - \xi|\},$$ 

(3)

then our non-degeneracy condition is the following: \( x_0 \) is a non-characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0,1) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0,T(x_0),\delta_0}. \quad (4)$$
It is an open problem to tell whether condition (4) holds for all space–time blow-up points. Let us recall our result about the blow-up rate valid also in higher dimensions under the condition

\[ N \geq 2 \quad \text{and} \quad 1 < p \leq p_c \equiv 1 + \frac{4}{N-1}. \]  

(5)

Given some \((x_0, T_0)\) such that \(0 < T_0 \leq T(x_0)\), we introduce the following self-similar change of variables:

\[ w_{x_0, T_0}(y, s) = (T_0 - t)^{\frac{2}{p-1}} u(x, t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \]  

(6)

If \(T_0 = T(x_0)\), then we simply write \(w_{x_0}\) instead of \(w_{x_0, T(x_0)}\). This change of variables transforms the backward light cone with vertex \((x_0, T_0)\) into the infinite cylinder \((y, s) \in B \times [-\log T_0, +\infty)\) where \(B = B(0, 1)\). The function \(w_{x_0, T_0}\) (we write \(w\) for simplicity) satisfies the following equation for all \(y \in B\) and \(s \geq -\log T_0:\)

\[ \partial_{ss}^2 w = Lw - 2\frac{p+1}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial_{ys}^2 w, \]  

(7)

where

\[ Lw = \frac{1}{\rho} \partial_y (\rho (1 - y^2) \partial_y w) \quad \text{and} \quad \rho(y) = (1 - y^2)^{\frac{2}{p-1}}. \]  

(8)

This equation will be studied in the space

\[ \mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1, 1) \mid \|q\|_{\mathcal{H}}^2 \equiv \int_{-1}^{1} (q_1^2 + (q_1')^2(1 - y^2) + q_2^2) \rho \, dy < +\infty \right\}, \]  

(9)

which is the energy space for \(w\). Note that \(\mathcal{H} = \mathcal{H}_0 \times L^2_{\rho}\) where

\[ \mathcal{H}_0 = \left\{ r \in H^1_{loc}(-1, 1) \mid \|r\|_{\mathcal{H}_0}^2 \equiv \int_{-1}^{1} (r^2(1 - y^2) + r^2) \rho \, dy < +\infty \right\}. \]  

(10)

This is the blow-up bound we obtain in [17] (see also [18, Proposition 2.2] for a statement):

**Uniform bounds on solutions of (7).** If \(u\) is a solution of (1) with blow-up surface \(\Gamma\): \(\{x \to T(x)\}\) and \(x_0 \in \mathbb{R}\), then for all \(s \geq -\log T(x_0) + 1\),

(E1) \(E(w_{x_0}(s)) \to E_\infty \geq 0\) as \(s \to \infty\).

(E2) There exists \(C_0 > 0\) such that for all \(s \geq s_0 + 1, \int_{-1}^{1} w_{x_0}(y, s)^2 \rho(y) \, dy \leq C_0\).

(E3) \(\int_{s}^{+\infty} \int_{-1}^{1} \frac{\partial_y w_{x_0}(y, s')^2}{1 - y^2} \rho(y) \, ds' \, dy \to 0\) as \(s \to \infty\).
There exists $C_0 > 0$ such that for all $s \geq s_0 + 1$,

$$
\int_1^{s+1} \int_{-1}^s \left\{ \partial_y w_{x_0}^2 (1 - y^2) + w_{x_0}^2 + \partial_s w_{x_0}^2 + |w_{x_0}|^{p+1} \right\} (y, s') \rho(y) \, dy \, ds' \leq C_0.
$$

If in addition $x_0$ is non-characteristic (in the sense (4)), then for all $s \geq -\log T(x_0) + 4$,

$$
0 < \epsilon_0(p) \leq \| w_{x_0}(s) \|_{H^1(-1,1)} + \| \partial_s w_{x_0}(s) \|_{L^2(-1,1)} \leq K, \quad (11)
$$

where $w_{x_0}$ is defined in (6) and $K$ depends only on $p$ and on an upper bound on $T(x_0)$, $1/T(x_0)$, $\delta_0(x_0)$ and the initial data in $H^1_{\text{loc}}, u \times L^2_{\text{loc}}, u$.

**Remark.** Note that the positivity of $E(w_{x_0}(s))$ is the only delicate point in making the analysis of [17] work for characteristic points. See Appendix A.

A natural question then is to know if $w_{x_0}(y, s)$ has a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).

In the context of Hamiltonian systems, this question is delicate, and there is no natural reason for such a convergence, since Eq. (1) is time reversible. See Martel and Merle [13] for the case of the $L^2$ critical Korteweg–de Vries equation, and Merle and Raphaël [14] for the case of the $L^2$ critical nonlinear Schrödinger equation.

For the case of the heat equation

$$
\partial_t u = \Delta u + |u|^{p-1}u, \quad (12)
$$

where $u: (x, t) \in \Omega \times [0, T) \to \mathbb{R}$ and $\Omega = \mathbb{R}^N$ or $\Omega$ is a bounded domain of $\mathbb{R}^N$, $p > 1$ and $(N - 2)p < N + 2$, the structure in self-similar variables is similar to that of the wave equation (1). However, the blow-up time $T$ is unique for Eq. (12). It is the time when the solution leaves the Cauchy space. What we call the blow-up set then is the set of all $x_0 \in \Omega$ such that $u(x, t)$ does not remain bounded as $(x, t)$ approaches $(x_0, T)$. Unlike the wave equation case, the blow-up set is a subset of $\mathbb{R}^N$ and not $\mathbb{R}^{N+1}$. As in (7), we can define a $w(y, s)$ in self-similar variables. We know from Giga and Kohn [6] that this $w(y, s)$ approaches a universal function (actually a constant), which turns to be the unique non-zero stationary solution (up to a sign change) in the self-similar variable. Note that in the heat equation case, the set of stationary solutions is made of three isolated solutions.

This paper is organized around two main results. We present each of them in a separate subsection.

**1.2. Convergence to the set of stationary solutions**

We first classify all $H^1_0$ stationary solutions of (7) in one dimension. More precisely, we prove the following proposition in Section 2.3.

**Proposition 1** (Classification of all stationary solutions of (7) in one dimension).
Consider \( w \in H_0 \) a stationary solution of (7). Then, either \( w \equiv 0 \) or there exist \( d \in (-1, 1) \) and \( \omega = \pm 1 \) such that \( w(y, s) = \omega \kappa(d, y) \) where
\[
\forall (d, y) \in (-1, 1)^2, \quad \kappa(d, y) = \kappa_0 \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^{\frac{2}{p-1}}} \quad \text{and} \quad \kappa_0 = \left( \frac{2(p + 1)}{(p - 1)^2} \right)^{\frac{1}{p-1}}. \tag{13}
\]

(ii) It holds that
\[
E(0) = 0 \quad \text{and} \quad \forall d \in (-1, 1), \quad E(\kappa(d, \cdot)) = E(-\kappa(d, \cdot)) = E(\kappa_0) > 0, \tag{14}
\]
where
\[
E(w(s)) = \int_{-1}^{1} \left( \frac{1}{2} (\partial_{s} w)^2 + \frac{1}{2} (\partial_{y} w)^2 (1 - y^2) + \frac{(p + 1)}{(p - 1)^2} w^2 - \frac{1}{p + 1} |w|^{p+1} \right) \rho(y) \, dy. \tag{15}
\]

**Remark.** Note that the set of stationary solutions consists of 3 connected components, one of them is the null singleton, and the two others are symmetric with respect to each other, and depend on one parameter. In the proof, we use the fact that \( N = 1 \). In higher dimensions, we are unable to classify all stationary solutions of (7) in \( H_0 \). Of course, we already know that \( \pm \kappa(d, \omega, y) \) is an \( H_0 \) stationary solution of (7) for any \( |d| < 1 \) and \( \omega \in \mathbb{R}^N \) with \( |\omega| = 1 \), but we are unable to say whether there are others or not. This missing information prevents us from extending our results to higher dimensions. Note that \( H^1 \subset H_0 \). Thus, the result holds in \( H^1 \) as well.

**Remark.** The functional \( E(w(s)) \) defined in (15) is a Lyapunov functional for Eq. (7). Indeed, we know from Antonini and Merle [4] that if \( w(y, s) \) is a solution to (7) defined for all \( (y, s) \in \mathbb{R} \times [s_1, s_2] \), then
\[
E(w(s_2)) - E(w(s_1)) = -\frac{4}{p - 1} \int_{s_1}^{s_2} \int_{-1}^{1} \left( \partial_{s} w(y, s) \right)^2 \frac{\rho(y)}{1 - y^2} \, dy \, ds. \tag{16}
\]

Then, we consider \( x_0 \in \mathbb{R} \) and show that \( w_{x_0}(y, s) \) defined in (6) approaches a non-null connected component of the stationary solutions’ set in the non-characteristic case, strongly in the \( H^1 \times L^2(-1, 1) \) norm, and in the characteristic case, a decoupled sum of stationary solutions. More precisely, we prove the following.

**Theorem 2 (Strong convergence related to the set of stationary solutions).** Consider \( u \) a solution of (1) with blow-up curve \( \Gamma : [x \rightarrow T(x)] \).

(A) Non-characteristic case. If \( x_0 \in \mathbb{R} \) is non-characteristic (in the sense (4)), then, there exists \( \omega^*(x_0) \in (-1, 1) \) such that:
(A.i) \( \inf_{|d| < 1} \| w_{x_0}(\cdot, s) - \omega^*(x_0)\kappa(d, \cdot) \|_{H^1(-1, 1)} + \| \partial_s w_{x_0} \|_{L^2(-1, 1)} \to 0 \) as \( s \to \infty \).
(A.ii) \( E(w_{x_0}(s)) \to E(\kappa_0) \) as \( s \to \infty \).

(B) Characteristic case. If \( x_0 \in \mathbb{R} \) is characteristic, then, there exist \( k(x_0) \in \mathbb{N} \), \( \omega_i^* = \pm 1 \) and continuous \( d_i(s) = \tanh \eta_i(s) \in (-1, 1) \) for \( i = 1, \ldots, k \) such that:
(B.i) \[ \| \left( \frac{u_{x_0}(s)}{\partial_s u_{x_0}(s)} \right) - \left( \sum_{i=1}^{k(x_0)} \omega_i^* \kappa(d(s), \cdot) \right) \|_H \to 0 \text{ as } s \to \infty. \]

(B.ii) \[ |\zeta_i(s) - \zeta_j(s)| \to \infty \text{ as } s \to \infty \text{ for } i \neq j. \]

(B.iii) \[ E(w_{x_0}(s)) \to k(x_0)E(\kappa_0) \text{ as } s \to \infty. \]

Remark. When \( k(x_0) = 0 \), the sum in (B.i) has to be understood as 0.

A natural question now in the non-characteristic case is to see whether \( w_{x_0}(s) \) converges to some \( \kappa(d^\infty(x_0)) \) as \( s \to \infty \) for a given \( d^\infty(x_0) \in (-1, 1) \) (in fact, with the method we use to answer this question, we treat also the characteristic case when \( k(x_0) = 1 \)). This question will be addressed in the next subsection.

1.3. Trapping near the set of non-zero stationary solutions

In this part, we work in the space \( \mathcal{H} \) defined in (9), which is a natural choice (the energy space in \( w \)). We consider \( w \in C([s^*, \infty), \mathcal{H}) \) a solution to Eq. (7), where \( w \) may be equal to \( w_{x_0} \) defined in (6) from \( u \), a blow-up solution to Eq. (1), with no restriction on \( x_0 \). In particular, \( x_0 \) may or may not be a characteristic point.

In the following, we show that if \( w(s^*) \) is close enough to some non-zero stationary solution and satisfies an energy barrier, then \( w(s) \) converges to a neighboring stationary solution as \( s \to \infty \). More precisely, we have the following.

**Theorem 3** (Trapping near the set of non-zero stationary solutions of (7)). There exist positive \( \epsilon_0 \), \( \mu_0 \) and \( C_0 \) such that if \( w \in C([s^*, \infty), \mathcal{H}) \) for some \( s^* \in \mathbb{R} \) is a solution of Eq. (7) such that

\[ \forall s \geq s^*, \quad E(w(s)) \geq E(\kappa_0), \tag{17} \]

and

\[ \left\| \left( \begin{array}{c} w(s^*) \\ \partial_s w(s^*) \end{array} \right) - \omega^* \left( \begin{array}{c} \kappa(d^*, \cdot) \\ 0 \end{array} \right) \right\|_\mathcal{H} \leq \epsilon^* \tag{18} \]

for some \( d^* \in (-1, 1) \), \( \omega^* = \pm 1 \) and \( \epsilon^* \in (0, \epsilon_0] \), where \( \mathcal{H} \) and its norm are defined in (9) and \( \kappa(d, y) \) in (13), then there exists \( d^\infty \in (-1, 1) \) such that

\[ |d^\infty - d^*| \leq C_0 \epsilon^*(1 - d^{*2}) \]

and for all \( s \geq s^* \):

\[ \left\| \left( \begin{array}{c} w(s) \\ \partial_s w(s) \end{array} \right) - \omega^* \left( \begin{array}{c} \kappa(d^\infty, \cdot) \\ 0 \end{array} \right) \right\|_\mathcal{H} \leq C_0 \epsilon^* e^{-\mu_0(s-s^*)}. \tag{19} \]

Remark. If \( w = w_{x_0} \) where \( x_0 \) is some non-characteristic point of \( u \), a blow-up solution to (1), one sees from Theorem 2(A.ii) and the monotonicity of the Lyapunov functional \( E(w) \) that condition (17) is already satisfied and can be dropped down from the statement of Theorem 3. More generally, when \( x_0 \) is characteristic and

\[ k(x_0) = 1, \tag{20} \]
we see from Theorem 2(B) that conditions (17) and (18) hold for \( s_0 \) large. In [20], we will see from Theorem 2 that (20) cannot occur with \( x_0 \) characteristic.

**Remark.** The condition (17) is necessary. Indeed, if the solution converges to some \( \kappa(d_\infty, \cdot) \), then we see from the monotonicity of the functional \( E(w(s)) \) that

\[
\forall s \geq s_0, \quad E(w(s)) \geq \lim_{s \to \infty} E(w(s)) = E(\kappa(d_\infty, \cdot)).
\]

Using (14), we see that (17) follows. In particular, the following function:

\[
w^*(y, s) = (1 + e^s)^{-\frac{2}{p-1}} \kappa \left( d, \frac{y}{1 + e^s} \right) = \kappa_0 \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + e^s + dy)^{\frac{2}{p-1}}}
\]

(which is a particular solution to (7); use (31) below) is a heteroclinic orbit connecting \( \kappa(d, \cdot) \) as \( s \to -\infty \) to 0 as \( s \to \infty \) and satisfies \( E(w^*(s)) < E(\kappa_0) \) for any \( s \in \mathbb{R} \).

**Remark.** Note that \( \epsilon_0 \) is independent of \( d^* \) in this theorem. This remarkable fact is very important in the characteristic case, as we show in a forthcoming paper [20]. One could think of using the Lorentz transform to reduce the analysis to the case \( d^* = 0 \), which would give a uniform \( \epsilon_0 \). This does not work, because the Lorentz transform mixes time and space. In our proof, we work uniformly in \( |d^*| < 1 \) in the space \( \mathcal{H} \) (9) which is well adapted to the measure of the distance between two solutions to Eq. (7), including in the characteristic case, and leads to exponential estimates.

Now, if \( w = w_{x_0} \) where \( x_0 \) is non-characteristic, then Theorems 2 and 3 apply (use (16) to derive (17) from (A.ii) in Theorem 2), and we obtain the convergence of \( w_{x_0} \) to some non-zero stationary solution in the norm of \( \mathcal{H} \). Using the uniform estimates (11), we directly get the following result.

**Corollary 4 (Blow-up profile near a non-characteristic point).** If \( u \) is a solution of (1) with blow-up curve \( \Gamma: \{ x \to T(x) \} \) and \( x_0 \in \mathbb{R} \) is non-characteristic (in the sense (4)), then there exist \( d_\infty(x_0) \in (-1, 1) \), \( |\omega^*(x_0)| = 1 \) and \( s^*(x_0) \geq -\log T(x_0) \) such that for all \( s \geq s^*(x_0) \), (19) holds with \( \epsilon^* = \epsilon_0 \), where \( C_0 \) and \( \epsilon_0 \) are given in Theorem 3. Moreover,

\[
\left\| w_{x_0}(s) - \omega^*(x_0) \kappa(d_\infty(x_0), y) \right\|_{H^1(-1, 1)} + \left\| \partial_s w_{x_0}(s) \right\|_{L^2(-1, 1)} \to 0 \quad \text{as} \ s \to \infty.
\]

**Remark.** The sign \( \omega^*(x_0) \) is given by Theorem 2. From condition (18) in Theorem 3, the time \( s^*(x_0) \) is completely explicit and characterized by the fact that

\[
s^*(x_0) = \inf_{s \geq -\log T(x_0)} \inf_{|r| < 1} \left\| \left( \frac{w(s)}{\partial_s w(s)} - \omega^*(x_0) \left( \kappa(d, \cdot), 0 \right) \right) \right\|_{\mathcal{H}} \leq \epsilon_0.
\]

**Remark.** Theorem 3 and Corollary 4 is a fundamental step towards new blow-up results by the authors in a new paper [20]. We prove there that the set of non-characteristic points \( I_0 \) is open and that \( \forall x \in I_0, T'(x) = d_\infty(x) \) defined in (19). This gives a geometrical interpretation for \( d_\infty(x) \) as the slope of the blow-up curve. For the moment, we are unable to prove this theorem in higher
dimensions. The main difficulty comes from the fact that we are unable to classify all $H^1$ stationary solutions of (7) in higher dimensions, even in the radially symmetric case. Nevertheless, we hope to carry this program in higher dimensions with the same approach, avoiding the lack of information on the stationary solutions by using some extra arguments.

This paper is organized as follows. In Section 2, we give some basic properties of Eq. (7) and prove Proposition 1 which characterizes the set of stationary solutions. In Section 3, we use energy methods to prove Theorem 2. Then, in Section 4, we study the linearized operator of Eq. (7) around a non-zero stationary solution. That study is far from being trivial, since this linearized operator is not self-adjoint. Finally, Section 5 is devoted to the proof of Theorem 3 (note that Corollary 4 is a direct consequence of Theorems 2 and 3). The proof of Theorem 3 is the most delicate part in the proof, because of the non-self-adjoint character of the linear operator, and because every non-zero stationary solution of (7) is not isolated. This difficulty will be overcome by using similar concepts to those used for the Korteweg–de Vries equation (Martel and Merle [13]) and the nonlinear Schrödinger equation (Merle and Raphaël [14]). See Section 5 for more details.

2. Preliminaries

This section is divided in 3 subsections.

In Section 2.1, we give some dispersive estimates of Eq. (7).

In Section 2.2, we give some properties of the Lorentz transform which keeps Eq. (1) invariant.

In Section 2.3, we prove Proposition 1 which characterizes the set of stationary solutions.

2.1. Dispersive and spectral properties for Eq. (7)

We first recall from [4] the following result which gives the boundedness for $E$ and its variation:

**Proposition 2.1 (Boundedness of the Lyapunov functional for Eq. (7)).**

(i) Consider $w(y,s)$, a solution to (7) defined for all $(y,s) \in (-1,1) \times [-\log T, +\infty)$ such that $(w, \partial_s w)(-\log T) \in H^1 \times L^2(-1,1)$. For all $s \geq -\log T$, we have

$$0 \leq E(w(s)) \leq E(w(-\log T))$$

and

$$\int_{-\log T}^{1} \int_{-1}^{1} (\partial_s w(y,s))^2 \frac{\rho(y)}{1-y^2} dy \, ds \leq \frac{p-1}{4} E(w(-\log T)).$$

**Remark.** Note that with this proposition, the analysis of [17] extends immediately to the case where $w = w_{x_0}$ with $x_0$ characteristic, and the estimates (E1)–(E4) of p. 45 are fully justified.

**Proof of Proposition 2.1.** See Antonini and Merle [4] and Appendix A. □

In the following, we give Hardy–Sobolev identities in the space $H_0$ (10).
Lemma 2.2 (A Hardy–Sobolev type identity). For all $h \in \mathcal{H}_0$, it holds that

$$\left( \int_{-1}^{1} h(y)^2 \frac{\rho(y)}{1 - y^2} \, dy \right)^{1/2} \leq C \|h\|_{\mathcal{H}_0}, \quad (21)$$

$$\|h\|_{L^p+1} \leq C \|h\|_{\mathcal{H}_0}, \quad (22)$$

$$\|h(1 - y^2)^{1-p} \|_{L^\infty(-1,1)} \leq C \|h\|_{\mathcal{H}_0}. \quad (23)$$

Proof of (21). Let us recall from [17] the following Hardy type inequality:

$$\int_{-1}^{1} h(y)^2 \frac{\rho(y)}{1 - y^2} \, dy \leq C \int_{-1}^{1} h(y)^2 \rho(y) + C \int_{-1}^{1} (h'(y))^2 (1 - y^2) \rho(y) = C \|h\|^2_{\mathcal{H}_0}$$

(see the appendix in [17] for a proof). Using the fact that $\frac{\rho(y)}{1 - y^2} = \rho + y^2 \frac{\rho(y)}{1 - y^2}$, we get (21). □

Proof of (22) and (23). Let us use the following change of variables:

$$\xi = \frac{1}{2} \log \left( \frac{1 + y}{1 - y} \right) \quad \text{(that is } y = \tanh \xi \text{) and } \tilde{h}(\xi) = h(y)(1 - y^2)^{1-p}.$$

Then,

$$\int_{-1}^{1} h(y)^{p+1} \rho(y) \, dy = \int_{-1}^{1} \tilde{h}(\xi)^{p+1} \frac{d\xi}{1 - y^2} = \int_{\mathbb{R}} \tilde{h}(\xi)^{p+1} \, d\xi \leq C_0 \left( \int_{-1}^{1} (\tilde{h}^2 + \tilde{h}_\xi^2) \, d\xi \right)^{\frac{p+1}{2}},$$

$$\|h(1 - y^2)^{1-p} \|_{L^\infty(-1,1)} = \|\tilde{h} \|_{L^\infty(\mathbb{R})} \leq C_0 \left( \int_{-1}^{1} (\tilde{h}^2 + \tilde{h}_\xi^2) \, d\xi \right)^{\frac{1}{2}}. \quad (25)$$

Note from (21) that

$$\int_{\mathbb{R}} \tilde{h}(\xi)^2 \, d\xi = \int_{-1}^{1} h(y)^2 \rho(y) \, d\xi = \int_{-1}^{1} h(y)^2 \rho(y) \, dy \leq C_0 \|h\|^2_{\mathcal{H}_0}, \quad (24)$$

$$\int_{\mathbb{R}} \tilde{h}_\xi(\xi)^2 \, d\xi \leq C_0 \left( \int_{-1}^{1} h(y)(1 - y^2) \rho(y) \, dy + \int_{-1}^{1} h(y)^2 \rho(y) \, dy \right) \leq C_0 \|h\|^2_{\mathcal{H}_0}, \quad (25)$$

which concludes the proof of (22) and (23) and Lemma 2.2. □
The Legendre operator
\[
L_w = \frac{1}{\rho} \partial_y (\rho (1 - y^2) \partial_y w), \quad \text{where } \rho(y) = (1 - y^2)^{\frac{p}{p-1}},
\]
involved in the expression of Eq. (7) has the following properties.

**Proposition 2.3** (Properties of the operator \(L\) (8)). The operator \(L\) is self-adjoint in \(L^2_\rho\). For each \(n \in \mathbb{N}\), there exists a polynomial \(h_n\) of degree \(n\) such that
\[
L h_n = \gamma_n h_n, \quad \text{where } \gamma_n = -n \left( n + \frac{p+3}{p-1} \right).
\]

The family \(\{h_n | n \in \mathbb{N}\}\) is orthonormal and spans the whole space \(L^2_\rho\). When \(n = 0\) and \(n = 1\), the eigenfunctions are \(h_0 = c_0\) and \(h_1 = c_1 y\) for some positive \(c_0\) and \(c_1\), and
\[
L c_0 = 0, \quad L c_1 y = -\frac{2(p+1)}{p-1} c_1 y.
\]

**Proof.** The proof is straightforward and classical. One can show that for some positive \(c_n\), \(h_n = \frac{c_n}{\rho} \frac{d^n}{dy^n} (\rho (1 - y^2)^n)\). \(\square\)

We claim the following.

**Lemma 2.4.** Consider \(u \in L^2_\rho\) such that \(L u \in L^2_\rho\) and
\[
\int_{-1}^{1} u(y) \rho(y) dy = \int_{-1}^{1} u(y) y \rho(y) dy = 0.
\]

Then,
\[
\int_{-1}^{1} u L u \rho dy \leq \gamma_2 \int_{-1}^{1} u^2 \rho dy \quad \text{where } \gamma_2 = -2 \frac{(3p+1)}{p-1}.
\]

**Proof.** From (28) and (27), we have
\[
\tilde{u}_0 = \tilde{u}_1 = 0, \quad \text{where } \tilde{u}_n = \int_{-1}^{1} u h_n \rho dy.
\]

Therefore, using (26), we write \(u = \sum_{n=2}^{\infty} \tilde{u}_n h_n\) and \(L u = \sum_{n=2}^{\infty} \gamma_n \tilde{u}_n h_n\). Using the orthogonality of the polynomials \(h_k\) and the fact that \(\gamma_n \leq \gamma_2\) for all \(n \geq 2\), we write
\[
\int_{-1}^{1} u L u \rho dy = \sum_{n=2}^{\infty} \gamma_n \tilde{u}_n^2 \leq \gamma_2 \sum_{n=2}^{\infty} \tilde{u}_n^2 = \gamma_2 \int_{-1}^{1} u^2 \rho dy.
\]

This concludes the proof of Lemma 2.4. \(\square\)
2.2. Invariance of Eq. (7)

In this section, we consider \( u(x, t) \) a solution of (1) defined in the cone

\[
\{(\xi, \tau) \mid t_1 \leq \tau < t_0 - |\xi - x_0|\}
\]

(30)

for some \( t_1 < t_0 \) and \( x_0 \in \mathbb{R} \). Using the transformation (7), we see that \( w = w_{x_0, t_0} \) is a solution of (7) defined for all \( |y| < 1 \) and \( s \in [-\log(t_0 - t_1), +\infty) \). Equation (1) is invariant under translations in time and space, scaling and the Lorentz transformation. Through the self-similar transformation (7), this provides us with 4 invariant transformations for Eq. (7). More precisely, the following transformations of \( w(y, s) \) are also solutions to (7):

- For any \( a \in \mathbb{R} \), the function \( w_1(y, s) \) defined for all \( s \in [-\log(t_0 - t_1), +\infty) \) and \( y \in (-ae^s - 1, -ae^s + 1) \) by
  \[
  w_1(y, s) = w(y + ae^s, s).
  \]

- For any \( b \leq t_0 - t_1 \), the function \( w_2(y, s) \) defined for all \( s \geq -\log(t_0 - t_1 - b) \) and \( |y| < 1 + be^s \) by
  \[
  w_2(y, s) = (1 + be^s)^{-\frac{2}{\nu - 1}} w\left(\frac{y}{1 + be^s}, s - \log(1 + be^s)\right).
  \]

- For any \( c \in \mathbb{R} \), the function \( w_3(y, s) \) defined for all \( |y| < 1 \) and \( s \in [-\log(t_0 - t_1) - c, +\infty) \) by
  \[
  w_3(y, s) = w(y, s + c).
  \]

- The transposition in self-similar variables of the Lorentz transform which will be given in this section.

Let us recall the invariance of Eq. (1) under the Lorentz transform.

**Lemma 2.5 (Invariance of Eq. (1) under the Lorentz transform).**

(i) Consider \( u(x, t) \) a solution of Eq. (1) defined in the cone (30). For any \( d \in (-1, 1) \), the function \( U \equiv Z_d(u) \) defined by

\[
U(x', t') = u(x, t), \quad \text{where} \quad x' = \frac{x + dt}{\sqrt{1 - d^2}} \quad \text{and} \quad t' = \frac{t + dx}{\sqrt{1 - d^2}}
\]

is also a solution of (1) defined in the set

\[
\{(x', t') \mid t_1 \sqrt{1 - d^2} + dx' \leq t' < t_0 + dx_0 - |x' - x_0'|\}, \quad \text{where}
\]

\[
x_0' = \frac{x_0 + dt_0}{\sqrt{1 - d^2}} \quad \text{and} \quad t_0' = \frac{t_0 + dx_0}{\sqrt{1 - d^2}}.
\]
(ii) For all \( d_1 \) and \( d_2 \) in \((-1,1)\), we have
\[
  d_1 \ast d_2 = \frac{d_1 + d_2}{1 + d_1 d_2}.
\]  

**Remark.** From (ii) of this proposition, we deduce that \( Z_d \circ Z_{-d} = Z_0 = \text{Id} \) for all \( d \in (-1,1) \).

**Proof.** Everything is straightforward, except maybe for the composition identity. Consider then \( d_1, d_2 \in (-1,1) \) and define
\[
  U = Z_{d_1} u \quad \text{by} \quad U(x', t') = u(x, t), \quad \text{where} \quad x' = \frac{x + d_1 t}{\sqrt{1 - d_1^2}} \quad \text{and} \quad t' = \frac{t + d_1 x}{\sqrt{1 - d_1^2}},
\]
and
\[
  \mathcal{U} = Z_{d_2} U \quad \text{by} \quad \mathcal{U}(x'', t'') = U(x', t'), \quad \text{where} \quad x'' = \frac{x' + d_2 t'}{\sqrt{1 - d_2^2}} \quad \text{and} \quad t'' = \frac{t' + d_2 x'}{\sqrt{1 - d_2^2}}.
\]
Then,
\[
x'' = \frac{x' + d_2 t'}{\sqrt{1 - d_2^2}} = \frac{x + d_1 t + d_2 (t + d_1 x)}{\sqrt{(1 - d_2^2)(1 - d_1^2)}} = \frac{x + t \frac{d_1 + d_2}{1 + d_1 d_2}}{\sqrt{(1 - d_2^2)(1 - d_1^2)}} = \frac{x + t (d_1 \ast d_2)}{\sqrt{1 - (d_1 \ast d_2)^2}}
\]
since
\[
  \frac{(1 - d_2^2)(1 - d_1^2)}{(1 + d_2 d_1)^2} = 1 - \left( d_1 + d_2 \frac{1 + d_1 d_2}{1 + d_2 d_1} \right)^2.
\]
Similarly, we have \( t'' = (t + x(d_1 \ast d_2))/\sqrt{1 - (d_1 \ast d_2)^2} \). Since \( \mathcal{U}(x'', t'') = U(x', t') = u(x, t) \), this implies that \( Z_{d_1} \circ Z_{d_2} = Z_{d_1 \ast d_2} \). \( \square \)

Through the self-similar transformation (6), the Lorentz transform provides a one-dimensional group which keeps invariant Eq. (7). More precisely,

**Lemma 2.6 (The Lorentz transform in similarity variables).** Consider \( w(y, s) \), a solution of Eq. (1) defined for all \( |y| < 1 \), and \( s \in (s_0, s_1) \) for some \( s_0 \) and \( s_1 \) in \( \mathbb{R} \), and introduce for any \( d \in (-1,1) \), the function \( W = T_d(w) \) defined by
\[
  W(Y, S) = \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dY)^{\frac{1}{p-1}}} w(y, s), \quad \text{where} \quad y = \frac{Y + d}{1 + dY} \quad \text{and} \quad s = S - \log \frac{1 + dY}{\sqrt{1 - d^2}}.
\] (33)
Then \( W(Y, S) = T_d(w) \) is also a solution of (7) defined for all \( |Y| < 1 \) and
\[
S \in \left( s_0 + \frac{1}{2} \log \frac{1 + |d|}{1 - |d|}, s_1 - \frac{1}{2} \log \frac{1 + |d|}{1 - |d|} \right).
\]

**Remark.** From (ii) in Lemma 2.5, we have \( T_{d_1} \circ T_{d_2} = T_{d_1 \ast d_2} \) and \( T_d \circ T_{-d} = T_0 = \text{Id} \) where the law \( \ast \) is defined in (32).

**Remark.** If \( w(y) \) is a stationary solution of (7), then the function \( W(Y) = T_d(w) \) depends only on \( Y \) and is also a stationary solution of (7).

**Proof.** Note that the domain of definition of \( W(Y, S) \) follows directly from (33). It remains to check that it is a solution to (7).

Let us define \( \tilde{W}(\tilde{Y}, \tilde{S}) \) by
\[
\tilde{W}(\tilde{Y}, \tilde{S}) = \left( t_0 - t' \right)^{2 \frac{2}{p-1}} U(x', t'), \quad \tilde{Y} = \frac{x' - x_0}{t_0 - t'} \quad \text{and} \quad \tilde{S} = -\log(t_0 - t'),
\]
where
\[
x_0 = \frac{d}{\sqrt{1 - d^2}}, \quad t_0 = \frac{1}{\sqrt{1 - d^2}},
\]
\[
U(x', t') = u(x, t), \quad x' = \frac{x + dt}{\sqrt{1 - d^2}}, \quad t' = \frac{t + dx}{\sqrt{1 - d^2}},
\]
\[
u(x, t) = (1 - t)^{-\frac{2}{p-1}} w(y, s), \quad y = \frac{x}{1 - t} \quad \text{and} \quad s = -\log(1 - t).
\]

Using the self-similar transformation (6), the Lorentz transform (36) and then again (6), we see that \( u \) and \( U \) are solutions to (1), and then \( \tilde{W}(\tilde{Y}, \tilde{S}) \) is a solution to (7). In the following, we will prove that \( \tilde{W} = W, \tilde{Y} = Y \) and \( \tilde{S} = S \), which will conclude the proof. Using (37) and (34), we write
\[
x = ye^{-s}, \quad t = 1 - e^{-s}, \quad x' = x_0 + \tilde{Y} e^{-\tilde{S}}, \quad t' = t_0 - e^{-\tilde{S}},
\]
\[
\tilde{W}(\tilde{Y}, \tilde{S}) = e^{-\frac{2s}{p-1}} U(x', t') \quad \text{and} \quad w(y, s) = e^{-\frac{2s}{p-1}} u(x, t).
\]

Using the Lorentz transform (36), we write
\[
\tilde{W}(\tilde{Y}, \tilde{S}) = e^{2 \frac{s - \tilde{S}}{p-1}} w(y, s), \quad \tilde{Y} e^{-\tilde{S}} + x_0 = \frac{ye^{-s} + d(1 - e^{-s})}{\sqrt{1 - d^2}},
\]
\[
t_0 - e^{-\tilde{S}} = \frac{1 - e^{-s} + dy e^{-s}}{\sqrt{1 - d^2}}.
\]
Using (35), this gives
\[
\tilde{S} = s - \log \frac{1 - dy}{\sqrt{1 - d^2}}, \quad \tilde{Y} = \frac{y - d}{1 - dy} \quad \text{and} \quad \tilde{W}(\tilde{Y}, \tilde{S}) = \frac{(1 - dy)^{\frac{2}{p-1}}}{(1 - d^2)^{\frac{1}{p-1}}} w(y, s).
\]
Therefore,

\[(1 - dy)(1 + d\tilde{Y}) = 1 - d^2, \quad y = \frac{\tilde{Y} + d}{1 + d\tilde{Y}} \quad \text{and} \quad \frac{1 - dy}{\sqrt{1 - d^2}} = \frac{\sqrt{1 - d^2}}{1 + d\tilde{Y}}.\]

Thus, using (33) and (39), we see that \(\tilde{W} = W, \tilde{Y} = Y\) and \(\tilde{S} = S\). Since \(\tilde{W}(\tilde{Y}, \tilde{S})\) is a solution to (7), the same holds for \(W(Y, S)\). This concludes the proof of Lemma 2.6. \(\square\)

For further purpose, we need to understand precisely the effect of the transformation \(T_d\) defined in (33) on the operator \(Lw\) which appears in (7) (regardless of the fact that \(w\) is a solution of (7) or not). In (i) of the following lemma, we transform all the terms (linear and nonlinear) of Eq. (7). In (ii), we show that in fact, the linearized operator of Eq. (7) around the constant solution \(\kappa_0\) (13) transforms into the linearized operator of the same equation around \(\kappa(d, y)\), the transformation of \(\kappa_0\) by the Lorentz transformation in similarity variables. More precisely, we claim the following.

**Lemma 2.7 (Transformations of the linearized operator of (7) around \(\kappa_0\)).** Consider a general \(w(y, s)\) not necessarily a solution to (7) and \(W = T_dw\) defined in (33). Then, it holds that:

(i) (Nonlinear version)

\[
\partial_{ss}^2 w - \left( Lw - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y\partial_{y,s}^2 w \right) = \left(1 + dY\right)^{\frac{2p}{p-1}} \left( \partial_{SS}^2 W - \left( LW + \psi(d,Y)W - \frac{p+3}{p-1} \partial_S W - 2Y\partial_{Y,S}^2 W \right) \right).
\]

(ii) (The linearized operator around \(\kappa_0\))

\[
\partial_{ss}^2 w - \left( Lw + \frac{2(p+1)}{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y\partial_{y,s}^2 w \right) = \left(1 + dY\right)^{\frac{2p}{p-1}} \left( \partial_{SS}^2 W - \left( LW + \psi(d,Y)W - \frac{p+3}{p-1} \partial_S W - 2Y\partial_{Y,S}^2 W \right) \right),
\]

where

\[
\psi(d,Y) = p\kappa(d,Y)^{p-1} - \frac{2(p+1)}{(p-1)^2} = \frac{2(p+1)}{(p-1)^2} \left( p \frac{(1 - d^2)}{(1 + dY)^2} - 1 \right).
\]

**Remark.** If we consider \(w(y, s) = w(y)\), then it holds for \(W = T_dw\) that

\[
Lw(y) + \frac{2(p+1)}{p-1} w(y) = \left(1 + dY\right)^{\frac{2p}{p-1}} \left( LW(Y) + \psi(d,Y)W(Y) \right),
\]

where \(W \equiv T_dw\) is given in (33).
Proof of Lemma 2.7. (i) Using (37), (36) and (34), we write
\[
\partial_{ss}^2 w - \left( \mathcal{L} w - \frac{2(p + 1)}{(p - 1)^2} w + |w|^{p-1} w - \frac{p + 3}{p - 1} \partial_s w - 2y \partial_{y,s}^2 w \right)
\]
\[
= (1 - t)^\frac{2p}{p-1} \left( \partial_{tt}^2 u - \partial_{xx}^2 u - |u|^{p-1} u \right)
\]
\[
= (1 - t)^\frac{2p}{p-1} \left( \partial_{t't'}^2 U - \partial_{x'x'}^2 U - |U|^{p-1} U \right)
\]
\[
= \left( \frac{1 - t}{t_0 - t'} \right)^\frac{2p}{p-1} \left[ \partial_{ss}^2 W - \left( \mathcal{L} W - \frac{2(p + 1)}{(p - 1)^2} W + |W|^{p-1} W - \frac{p + 3}{p - 1} \partial_s W - 2y \partial_{y,s}^2 W \right) \right].
\]
(43)

Using (36), we see that \( t = (t' - dx')/\sqrt{1 - d^2} \). Therefore, using (37) and (35), we write
\[
\frac{1 - t}{t_0 - t'} = \frac{1 - \frac{t' - dx'}{\sqrt{1 - d^2}}}{t_0 - t'} = \frac{t_0 - t' + d(x' - x_0)}{(t_0 - t')\sqrt{1 - d^2}} = \frac{1 + dY}{\sqrt{1 - d^2}}.
\]
Using (43), this concludes the proof of (i) of Lemma 2.7.

(ii) Using (33), we write
\[
p^\frac{2(p + 1)}{(p - 1)^2} w - |w|^{p-1} w = \left( \frac{1 + dY}{1 - d^2} \right)^\frac{2p}{p-1} \left( \frac{2(p + 1)}{(p - 1)^2} W + \frac{p + 3}{p - 1} \partial_s W - 2Y \partial_{Y,s}^2 W \right),
\]
which shows the same factor as in (40). Subtracting this from (40), we get the conclusion of Lemma 2.7. □

In the following, we show that the transformation defined in (33) is continuous from \( \mathcal{H}_0 \) to \( \mathcal{H}_0 \) defined in (10).

Lemma 2.8 (Continuity of \( T_d \) in \( \mathcal{H}_0 \)). There exists \( C_0 > 0 \) such that for all \( d \in (-1, 1) \) and \( w \in \mathcal{H}_0 \), we have
\[
\frac{1}{C_0} \|w\|_{\mathcal{H}_0} \leq \|T_d(w)\|_{\mathcal{H}_0} \leq C_0 \|w\|_{\mathcal{H}_0}.
\]
(44)

Proof. We only prove the second inequality of (44), since the first one follows by applying the second one to \( T_{-d}(w) \) and using the fact that \( T_d \circ T_{-d} = \text{Id} \) (see the remark following Lemma 2.6).

If we consider \( W = T_d w \) defined in (33), then we see that
\[
\partial_{Y} W(Y) = -\frac{2d}{p - 1} \left( \frac{1 - d^2}{1 + dY} \right)^\frac{1}{p-1} w(y) + \left( \frac{1 - d^2}{1 + dY} \right)^\frac{1}{p-1} \frac{1}{1 + dY} \partial_{Y} w(y), \quad \text{where} \quad y = \frac{Y + d}{1 + dY}.
\]
Using (10) and (33), we write
Performing the change of variables $y = \frac{Y + d}{1 + dY}$, we get

$$\|W\|_{\mathcal{H}_0}^2 \leq C \int_{-1}^{1} \left(1 - y^2\right)^{\frac{2}{p-1}} w(y)^2 \left(\frac{1 - d^2}{(1 - dy)^2}\right) dy + C \int_{-1}^{1} \left(1 - y^2\right)^{\frac{2}{p-1} + 1} w(y)^2 \frac{1}{(1 - dy)^2} dy$$

$$+ C \int_{-1}^{1} \left(1 - y^2\right)^{\frac{2}{p-1} + 1} \left(\partial_y w(y)\right)^2 dy. \quad (45)$$

Using the fact that

$$\forall (d, y) \in (-1, 1)^2, \quad |y + d| + |1 - d^2| + (1 - y^2) \leq C(1 + dy), \quad (46)$$

and (21), we see that

$$\|W\|_{\mathcal{H}_0}^2 \leq \int_{-1}^{1} \left(1 - y^2\right)^{\frac{2}{p-1}} w(y)^2 dy + C \|w\|_{\mathcal{H}_0}^2 \leq C \|w\|_{\mathcal{H}_0}^2$$

and the conclusion follows. ⊓⊔

### 2.3. Characterization of the stationary solutions in self-similar variables

In this section, we prove Proposition 1 which characterizes all $\mathcal{H}_0$ solutions of

$$\frac{1}{\rho} \left( \rho (1 - y^2) w' \right)' - \frac{2(p + 1)}{(p - 1)^2} w + |w|^{p - 1} w = 0, \quad (47)$$

the stationary version of (7). Note that since 0 and $\pm \kappa_0$ are trivial solutions to Eq. (7), we see from Lemma 2.6 that $\pm T_d \kappa_0 = \pm \kappa(d, y)$ are also stationary solutions to (7). Let us introduce the set

$$S \equiv \{0, \kappa(d, \cdot), -\kappa(d, \cdot) \mid |d| < 1\}. \quad (48)$$
Now, we prove Proposition 1 which states that there are no more solutions of (47) in $\mathcal{H}_0$ other than the set $S$. We first prove (ii) since it is shorter and then prove (i).

(ii) Since we clearly have from the definition (15) of $E(\kappa(d, \cdot))$ that $E(0) = 0$, we only compute $E(\pm \kappa(d, \cdot))$. Since $\kappa(d, y)$ is a solution to Eq. (47), we multiply the equation by $\kappa(d, y)\rho(y)$ and integrate it with respect to $y \in (-1, 1)$ to get

$$-\int_{-1}^{1} |\partial_y \kappa(d, y)|^2 (1 - y^2) \rho(y) - \frac{2(p + 1)}{(p - 1)^2} \int_{-1}^{1} \kappa(d, y)^2 \rho(y) dy + \int_{-1}^{1} \kappa(d, y)^p + 1 \rho(y) dy = 0.$$  

Therefore, we see from (15) that $E(\kappa(d, \cdot)) = \frac{p - 1}{2(p + 1)} \int_{-1}^{1} \kappa(d, y)^p + 1 \rho(y) dy$. Making the change of variables $Y = \frac{y + d}{1 + dy}$, we see that

$$E(\kappa(d, \cdot)) = \frac{p - 1}{2(p + 1)} \int_{-1}^{1} \kappa(d, y)^p + 1 \rho(y) dy = \frac{p - 1}{2(p + 1)} \kappa_0^{p + 1} \int_{-1}^{1} \rho(Y) dY = E(\kappa_0) > 0,$$

$$\frac{1}{2} \int_{-1}^{1} |\partial_y \kappa(d, y)|^2 (1 - y^2) \rho(y) + \frac{(p + 1)}{(p - 1)^2} \int_{-1}^{1} \kappa(d, y)^2 \rho(y) dy = \frac{p + 1}{p - 1} E(\kappa_0). \quad (49)$$

Thus, (14) follows.

(i) Consider $w \in \mathcal{H}_0$ a non-zero solution of (47). Let us prove that there is some $d \in (-1, 1)$ such that $w = \pm \kappa(d, \cdot)$. For this purpose, consider

$$\xi = \frac{1}{2} \log \left( \frac{1 + y}{1 - y} \right) \quad \text{(that is } y = \tanh \xi \text{) and } \tilde{w}(\xi) = w(y) (1 - y^2)^\frac{1}{p - 1}. \quad (50)$$

Remark first from (24) and (25) that $\tilde{w} \in H^1(\mathbb{R})$. Let us prove that if $w \not\equiv 0$ is solution to (47) then it is equivalent to $\tilde{w} \not\equiv 0$ which is a solution to

$$\tilde{w}_{\xi\xi} + |\tilde{w}|^{p - 1} \tilde{w} - \frac{4}{(p - 1)^2} \tilde{w} = 0. \quad (51)$$

Indeed, we have

$$\tilde{w}_\xi = -\frac{2y}{p - 1} (1 - y^2)^\frac{1}{p - 1} w + w_y (1 - y^2)^\frac{1}{p - 1} + 1,$$

$$\tilde{w}_{\xi\xi} = \left[ -\frac{2}{p - 1} y (1 - y^2)^\frac{1}{p - 1} \right] w - \frac{2}{p - 1} y (1 - y^2)^\frac{1}{p - 1} + 1 w_y$$

$$- \frac{2y p}{p - 1} (1 - y^2)^\frac{1}{p - 1} + 1 w_y + w_{yy} (1 - y^2)^\frac{1}{p - 1} (1 - y^2)^2$$

$$= \left[ -\frac{2 (1 - y^2)}{p - 1} + \frac{4y^2}{(p - 1)^2} \right] w - \frac{2(p + 1)}{p - 1} y w_y (1 - y^2) + w_{yy} (1 - y^2)^2 \right] (1 - y^2)^\frac{1}{p - 1}. \quad (52)$$
Thus,
\[
\ddot{w}_{\xi\xi} - \frac{4}{(p-1)^2} \dot{w} + |\dot{w}|^{p-1} \dot{w} = (1 - y^2)^{1+\frac{1}{p-1}} \left[ -2 \frac{(p+1)}{(p-1)^2} w - \frac{2(p+1)}{p-1} y w_y + w_{yy}(1 - y^2) + |w|^{p-1} w \right]
\]
which proves the equivalence.

It is classical that all non-zero solutions of (51) in \(H^1(\mathbb{R})\) have the form
\[
\ddot{w}(\xi) = \pm \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi + \xi_0)} \quad \text{for} \quad \xi_0 \in \mathbb{R}.
\]
Thus, for \(d = \tanh \xi_0 \in (-1, 1)\) and \(y = \tanh \xi\), we write:
\[
\ddot{w}(\xi) = \pm \kappa_0 \left[ 1 - \tanh(\xi + \xi_0)^2 \right]^{\frac{1}{p-1}} = \pm \kappa_0 \left[ 1 - \left( \frac{\tanh \xi + \tanh \xi_0}{1 + \tanh \xi \tanh \xi_0} \right)^2 \right]^{\frac{1}{p-1}}
\]
\[
= \pm \kappa_0 \left[ 1 - \left( \frac{y + d}{1 + dy} \right)^2 \right]^{\frac{1}{p-1}} = \pm \kappa_0 \left[ \frac{(1 - d^2)(1 - y^2)}{(1 + dy)^2} \right]^{\frac{1}{p-1}} = \pm \kappa(d, y)(1 - y^2)^{\frac{1}{p-1}}.
\]
This means by (50) that \(w(y) = \pm \kappa(d, y)\), which concludes the proof of Proposition 1.

3. Energy estimates and convergence to the set of stationary solutions

This section is devoted to the proof of Theorem 2. In a pedagogical approach, we treat the non-characteristic case first, and then the general case. Indeed, in this first case, we will replace the use of an averaging property of the equation (useful in the general case) by the use of the finite speed of propagation.

3.1. The non-characteristic case

We prove Theorem 2(A) in this section. Note first that using the continuity of the Lyapunov functional \(E(w)\) (15) in the space \(H^1 \times L^2(-1, 1)\) and (14), (A.ii) directly follows from (A.i). Thus, we only prove (A.i). Consider a non-characteristic point \(x_0 \in \mathbb{R}\) and introduce
\[
w = w_{x_0} = w_{x_0,T(x_0)}.
\]
From (11) (proved in [18]), the Sobolev injection and Proposition 2.1, we have the following bounds.

**Lemma 3.1.** (Boundedness of \(w(s)\), see [18].) There exists \(K > 0\) such that for all \(s \geq -\log \frac{T(x_0)}{4}\),
\[
0 < \epsilon_0(p) \leq \|w(s)\|_{H^1(-1,1)} + \|\partial_s w(s)\|_{L^2(-1,1)} \leq K, \quad \|w(s)\|_{L^\infty(-1,1)} \leq K.
\]
and

\[ \int_{-\log T(x_0) - 1}^{\infty} \int_{-1}^{1} \left( \partial_y w(y, s) \right)^2 \frac{\rho(y)}{1 - y^2} dy \, ds \leq K. \quad (56) \]

We will show that there exists \( \omega(x_0) \in \{-1, 1\} \) such that

\[ \inf_{|d| < 1} \left\| w(\cdot, s) - \omega(x_0)\kappa(d, \cdot) \right\|_{H^1(-1, 1)} + \left\| \partial_s w \right\|_{L^2(-1, 1)} \to 0 \quad \text{as } s \to \infty. \quad (57) \]

It is a remarkable fact for a dispersive equation that a solution converges strongly to a stationary solution (as in the case of a dissipative equation). We first have the following reduction.

**Proposition 3.2.** In order to prove (57), it is enough to prove that

\[ \inf_{\tilde{w} \in S} \left\| w(s) - \tilde{w} \right\|_{H^1(-1, 1)} + \left\| \partial_s w \right\|_{L^2(-1, 1)} \to 0 \quad \text{as } s \to \infty, \quad (58) \]

where \( S \) is the set of all \( \mathcal{H}_0 \) stationary solutions to (7).

**Proof.** From Proposition 1 and (48), we know that \( S = S_1 \cup S_2 \cup \{0\} \) where \( S_1 = \{ \kappa(d, \cdot) \mid |d| < 1 \} \) and \( S_2 = \{-\kappa(d, \cdot) \mid |d| < 1 \} \). From the Sobolev injection, positivity and (13), we have for \( i = 1, 2 \):

\[ d_{H^1(-1, 1)}(S_i, 0) \geq C d_{L^\infty(-1, 1)}(S_i, 0) \geq C \inf_{|d| < 1} \left\| \kappa(d, \cdot) \right\|_{L^\infty(-1, 1)} \geq C_0 > 0. \]

\[ d_{H^1(-1, 1)}(S_1, S_2) \geq C d_{L^\infty(-1, 1)}(S_1, S_2) \geq C d_{L^\infty(-1, 1)}(S_1, 0) \geq C_0 > 0. \]

Since \( (w(s), \partial_s w(s)) \) is continuous as a function of \( s \) in \( H^1 \times L^2 \) and its norm is bounded from below by (54), we see that (58) implies (57). This concludes the proof of Proposition 3.2. \( \square \)

We now prove (58), which by Proposition 3.2 will conclude the proof of (57) and of Theorem 2(A). We proceed by contradiction and assume that there exist \( \epsilon_0 > 0 \) and a sequence \( s_n \to \infty \) such that

\[ \inf_{\tilde{w} \in S} \left\| w(s_n) - \tilde{w} \right\|_{H^1(-1, 1)} + \left\| \partial_s w(s_n) \right\|_{L^2(-1, 1)} \geq \epsilon_0 > 0. \quad (59) \]

We proceed in 2 steps.

- In Step 1, we show that \( w(s_n) \) converges in \( L^\infty \) to some \( w^* \in S \). This step will be a consequence of the existence of the Lyapunov functional \( E \) and compactness related to the uniform bounds we have in (54).
- In Step 2, using the space–time localization of the original energy for the function \( u(t) \), we find an estimate on \( w(s_n) \) which contradicts (59). This step is remarkable, in the setting of Hamiltonian systems (for example, this fact is false for \( L^2 \) critical NLS and \( L^2 \) critical KdV; see [14] and [13]).
Step 1. Convergence of $w(s_n)$ to a stationary solution in $L^\infty(-1, 1)$. By (54), there is a subsequence (still denoted by $s_n$) and $w^* \in H^1(-1, 1)$ such that

$$
\|w(s_n) - w^*\|_{L^\infty(-1, 1)} \to 0 \quad \text{as } n \to \infty.
$$

We have the following lemma.

Lemma 3.3.

(i) For any $M > 0$, we have

$$
w(y, s_n + s) - w^*(y) \to 0 \quad \text{as } n \to \infty, \text{ uniformly for } |y| < 1 \text{ and } |s| < M.
$$

(ii) We have $w^* \in S$.

Proof. (i) From (54) and (56), we have for all $M > 0$,

$$
\int_{|y| < 1 - \frac{1}{M}} |w(y, s_n + s) - w^*(y)|^2 \, dy 
\leq \int_{|y| < 1 - \frac{1}{M}} |w(y, s_n) - w^*(y)|^2 \, dy + C_0 \int_{s_n - M}^{s_n + M} \left( \int_{|y| < 1 - \frac{1}{M}} (\partial_s w(s_n + s', y))^2 \, dy \right)^{1/2} \, ds' 
\leq \int_{|y| < 1 - \frac{1}{M}} |w(y, s_n) - w^*(y)|^2 \, dy + C(M) \left( \int_{s_n - M - 1}^{s_n + M} \int_{|y| < 1 - \frac{1}{M}} (\partial_s w(s_n + s', y))^2 \, dy \rho \, ds' \right)^{1/2} \to 0
$$
as $n \to \infty$. From the fact that $\|v\|_{L^\infty(|y| < 1 - \frac{1}{M})} \leq C(M)\|v\|_{L^2(|y| < 1 - \frac{1}{M})}\|v\|_{H^1(|y| < 1 - \frac{1}{M})}$, we see that $w(y, s_n + s) - w^*(y) \to 0$ as $n \to \infty$, uniformly for $|y| < 1 - \frac{1}{M}$ and $|s| < M$. Since from (54) we have $\|w(y, s_n + s) - w^*(y)\|_{C^2((-1, 1))} \leq C_0$, (i) follows.

(ii) Here, we use the fact that $w(y, s)$ is a weak solution of (7), i.e. for any $C^\infty$ function $\varphi(y, s)$ compactly supported in $(-1, 1) \times (s_1, \infty)$ and some $s_1 \in \mathbb{R}$,

$$
I = \int \left( \mathcal{L}(\varphi)w - \frac{2(p + 1)}{(p - 1)^2} w^p \varphi + |w|^{p-1} w \varphi \right) \rho \, dy \, ds 
+ \int \partial_s w \left( \partial_s \varphi - \frac{p + 3}{p - 1} \varphi + \frac{1}{\rho} \partial_s (2y \varphi) \right) \rho \, dy \, ds = 0
$$

(see below for a proof of this fact).

For $\varphi_1(y) \in C^\infty$ compactly supported in $[-1 + \frac{1}{M}, 1 - \frac{1}{M}]$, consider $\varphi(y, s) = \varphi_1(y)\varphi_2(s - s_n)$ where $\varphi_2 \in C^\infty$, supp $\varphi_2 \in [-2, 2]$ and $\int_{\mathbb{R}} \varphi_2 = 1$ and apply (60).
Since
\[
\int_{s_n^{-2} \left| y \right| < 1 - \frac{1}{M}} \left( \partial_y w(y, s') \right)^2 dy ds' \to 0
\]
by (56), we use (i) of this lemma and the Cauchy–Schwarz inequality to get as \( n \to \infty \):
\[
\int_{-1}^{1} \left[ w^* \mathcal{L} \varphi_1 + \left( -\frac{2(p + 1)}{(p - 1)^2} w^* + \left| w^* \right|^{p-1} w^* \right) \varphi_1 \right] \rho \, dy = 0. \tag{61}
\]
Since \( w^* \in H^1(-1, 1) \), we obtain from classical elliptic regularity theory that \( w^* \in C^2(-1, 1) \), therefore, \( w^* \) satisfies Eq. (47), which is the conclusion of Lemma 3.3(ii). It remains to prove (60).

**Proof of (60).** Let us remark from the definition of \( w \) given in (6) that
\[
\partial_{ss}^2 w - \left( \mathcal{L} w - \frac{2(p + 1)}{(p - 1)^2} w + \left| w \right|^{p-1} w - \frac{p + 3}{p - 1} \partial_y w - 2y \partial_{yy}^2 w \right) = \frac{(\partial^2_{tt} u - \partial^2_{xx} u - \left| u \right|^{p-1} u)}{(T - t)^{-\frac{2p}{p-1}}},
\]
and thus for all \( C^\infty \) function \( \varphi(y, s) \) compactly supported in \((-1, 1) \times (s_1, \infty)\), for some \( s_1 \in \mathbb{R} \), we have
\[
I = \int_{C} \left( u \partial^2_{tt} \psi - u \partial^2_{xx} \psi - \left| u \right|^{p-1} u \psi \right) \, dx \, dt, \tag{62}
\]
where \( C = \{(x, t) \mid T - e^{s_1} < t < T, \ |x - x_0| < T - t \} \) and \( \psi(x, t) \) is \( C^\infty \) compactly supported in \( C \) and defined by \( \psi(x, t) = \varphi(y, s) e^{-\frac{2s}{p-1}} \rho(y) \), where \( y = \frac{x - x_0}{T - t} \) and \( s = -\log(T - t) \).

The Duhamel representation for \( u \) (where \( u_0 \in H^1_{\text{loc}} \) and \( u_1 \in L^2_{\text{loc}} \)):
\[
\begin{align*}
u(x, t) &= \frac{1}{2} \left( u_0(x + t) + u_0(x - t) \right) + \frac{1}{2} \int_{x - t}^{x + t} \frac{1}{2} \left( u_1 + \frac{1}{2} \int_{0}^{t} \int_{x - t + \tau}^{x + t - \tau} \left| u \right|^{p-1} u(z, \tau) \, dz \, d\tau \right) \, dx.
\end{align*}
\]
(63)

yields that \( u \) is also a weak solution of (1), hence, \( I = 0 \). Let us briefly recall the proof of this fact. Making the change of variables
\[
\tilde{u}(\xi, \eta) = u(x, t), \quad \tilde{\psi}(\xi, \eta) = \psi(x, t) \quad \text{with} \quad \xi = x + t \quad \text{and} \quad \eta = x - t,
\]
we write
\[
I = \frac{1}{2} \int \left( -4\tilde{u}(\xi, \eta) \partial^2_{\xi \eta} \tilde{\psi}(\xi, \eta) - \left| \tilde{u} \right|^{p-1} \tilde{u}(\xi, \eta) \tilde{\psi}(\xi, \eta) \right) \, d\xi \, d\eta,
\]
\[
\tilde{u}(\xi, \eta) = \frac{1}{2} \left( u_0(\xi) + u_0(\eta) \right) + \frac{1}{2} \int_{\eta}^{\xi} u_1 + \frac{1}{2} \int_{\eta + \tau}^{\xi - \tau} \left| u \right|^{p-1} u(z, \tau) \, dz \, d\tau.
\]
Integrating by parts and using Fubini’s identity, we get

\[-4 \int \tilde{u}(\xi, \eta) \partial_{\xi}^2 \tilde{\psi}(\xi, \eta) \, d\xi \, d\eta = 2 \int \left( \partial_{\xi} \int_{0}^{\frac{\xi - \eta}{2}} |u|^{p-1} u(z, \tau) \, dz \, d\tau \right) \partial_{\eta} \tilde{\psi}(\xi, \eta) \, d\xi \, d\eta \]

\[= 2 \int \left( \int_{0}^{\frac{\xi - \eta}{2}} |u|^{p-1} u(\xi - \tau, \tau) \, d\tau \right) \partial_{\eta} \tilde{\psi}(\xi, \eta) \, d\xi \, d\eta \]

\[= \int |\tilde{u}|^{p-1} u(\xi, \tau) \tilde{\psi}(\xi, \eta) \, d\xi \, d\eta.\]

Hence, \( I = 0 \) and (60) is proved. This concludes the proof of Lemma 3.3. \( \square \)

**Step 2.** \( H^1 \) control through the localization in the \( u \) variable. The following lemma allows us to conclude the proof of Theorem 2 in the non-characteristic case.

**Lemma 3.4.** For \( n \) large, we have

\[ \| w(s_n) - w^* \|_{H^1(-1,1)} + \| \partial_s w(s_n) \|_{L^2(-1,1)} \leq \frac{\epsilon_0}{2}, \]

where \( \epsilon_0 \) is defined in (59).

Indeed, taking \( n \) large, we have from this lemma a contradiction with (59), hence, (58) holds and by Proposition 3.2, (57) holds and so does Theorem 2 in the non-characteristic case.

**Proof of Lemma 3.4.** We claim it as a consequence of the localization of the energy in the \( u \) variable (finite speed of propagation) and the scaling factor coming from the self-similar transformation (6).

For \( B = B(\epsilon_0) > 0 \) to be chosen later large enough, consider

\[ W_n(y, s) = w(y, s + s_n - B). \]  

(64)

From (54) and the previous step, we know that for all \( n \in \mathbb{N} \):

- \( W_n \) and \( w^* \) are solutions to Eq. (7);
- for all \( s \geq 0 \), \( \| W_n(s) \|_{H^1(-1,1)} + \| \partial_s W_n(s) \|_{L^2(-1,1)} + \| w^* \|_{H^1(-1,1)} \leq C \);
- \( \sup_{s \in [0, B]} \| W_n(s) - w^* \|_{L^\infty(-1,1)} \leq \epsilon_n \to 0. \)  

(65)

Introducing \( u_n \) and \( u \) defined as in the self-similar transformation (7) by
\[ u_n(\xi, \tau) = (1 - \tau)^{-\frac{2}{p-1}} W_n \left( \frac{\xi}{1 - \tau}, -\log(1 - \tau) \right), \]

\[ u^*(\xi, \tau) = (1 - \tau)^{-\frac{2}{p-1}} w^* \left( \frac{\xi}{1 - \tau} \right), \tag{66} \]

we see that:

- \( u_n \) and \( u^* \) are solutions of (1) defined in \( \{(\xi, \tau) | 0 \leq \tau < 1 \text{ and } |\xi| < 1 - \tau\} \);
- \( \|u_n(0)\|_{H^1(-1,1)} + \|\partial_t u_n(0)\|_{L^2(-1,1)} + \|u^*(0)\|_{H^1(-1,1)} \leq C_0 \) (note that \( C_0 \) is independent from \( B \));
- \( \sup_{\tau \in [0, \tau_B]} \|u_n(\tau) - u^*(\tau)\|_{L^\infty(|\xi| < 1 - \tau)} = C(B)\epsilon_n \to 0 \) where \( \tau_B = 1 - e^{-B} \).

Consider for \( \tau \in [0, \tau_B] \), \( v_n(\tau) = u_n(\tau) - u^*(\tau) \). We have:

- \( (\partial_{\tau}^2 - \partial_{\xi}^2) v_n = f_n \) where \( \sup_{\tau \in [0, \tau_B]} \|f_n(\tau)\|_{L^\infty(|\xi| < 1 - \tau)} = C(B)\epsilon_n \to 0 \) as \( n \to \infty \);
- there is \( C_0 > 0 \) such that for all \( n \), \( I(0) \leq C_0 \), where

\[
I(\tau) = \int_{|\xi| < 1 - \tau} \left( (\partial_{\xi} v_n(\xi, \tau))^2 + (\partial_\tau v_n(\xi, \tau))^2 \right) d\xi.
\]

Let us prove that for \( n \) large, \( I(\tau_B) \leq 2C_0 \). Indeed, we have by a direct computation, for all \( \tau \in [0, \tau_B] \),

\[
I'(\tau) \leq 2 \int_{|\xi| < 1 - \tau} f_n \partial_\tau v_n(\xi, \tau) d\xi \leq C(B)\epsilon_n \sqrt{I(\tau)},
\]

which leads by integration in time for \( \epsilon_n \) small enough, to \( I(\tau_B) \leq 2C_0 \).

Note that we have from (66),

\[
\partial_\xi u_n(\xi, \tau) = (1 - \tau)^{-\frac{2}{p-1}} \partial_\xi W_n \left( \frac{\xi}{1 - \tau}, -\log(1 - \tau) \right),
\]

\[
\partial_\tau u_n(\xi, \tau) = (1 - \tau)^{-\frac{2}{p-1}} \left( \partial_\tau W_n + y \partial_y W_n + \frac{2}{p-1} W_n \right) \left( \frac{\xi}{1 - \tau}, -\log(1 - \tau) \right), \tag{67}
\]

and the same holds for \( u^* \). Using (66) and (67), we obtain

\[
\|\partial_j W_n(B) - \partial_j w^*(B)\|_{L^2(-1,1)} \leq e^{-\frac{2B}{p-1} - \frac{B}{p}} \|\partial_\xi v_n(\tau_B)\|_{L^2(|\xi| < \tau_B)} \leq C'_0 e^{-\frac{2B}{p-1} - \frac{B}{p}}, \tag{68}
\]

where \( C'_0 \) is independent from \( B \), and similarly, using (65)

\[
\|\partial_j W_n(B)\|_{L^2(-1,1)} \leq e^{-\frac{2B}{p-1} - \frac{B}{p}} \left( \|\partial_\tau v_n(\tau_B)\|_{L^2(|\xi| < \tau_B)} + \|\partial_\xi v_n(\tau_B)\|_{L^2(|\xi| < \tau_B)} \right) + \frac{2}{p-1} \|W_n(B) - w^*\|_{L^\infty(-1,1)} \leq C'_0 e^{-\frac{2B}{p-1}} + C\epsilon_n. \tag{69}
\]
Therefore, since $W_n(y, B) = w(y, s_n)$ by (64), we have from (65), (68) and (69),
\[
\|w(s_n) - w^*\|_{H^1(-1,1)} + \|\partial_s w(s_n)\|_{L^2(-1,1)} \leq C_0 e^{-\frac{2B}{p-1}} + C(B) \epsilon_n.
\]
Taking $B = B(\epsilon_0)$ and $n$ large enough, we get to the conclusion of Lemma 3.4. \hfill \Box

3.2. The characteristic case

Let us now consider $x_0$ a characteristic point and introduce $s_0 = -\log T(x_0)$. The known facts are limited in the characteristic case. Nevertheless, thanks to Appendix A, Section 2 of [17] applies and we know for $w = w_{x_0}$ that (E1)–(E4) in p. 45 hold.

Note that the proof we present works of course in the non-characteristic case also.

We proceed in two parts:

- In Part 1, we show that all the terms in the Lyapunov functional are bounded (Proposition 3.5), and then, we prove a local convergence result under a non-vanishing condition (Proposition 3.8).
- In Part 2, we conclude the proof of Theorem 2 in the characteristic case.

Part 1. Local convergence under a non-vanishing condition. Improving (E1)–(E4), we now claim that each term of the Lyapunov functional $E(w)$ is bounded separately.

**Proposition 3.5 (Boundedness of each term of $E(w)$ and convergence).**

(i) There is a $C_0 > 0$ such that for all $s \geq s_0 + \frac{3}{2}$,
\[
\int_{-1}^{1} (\partial_y w(s))^2 (1 - y^2) + w(s)^2 + \partial_s w(s)^2 + |w(s)|^{p+1} \rho \leq C_0.
\]

(ii) $\frac{1}{2} \int_{-1}^{1} \partial_y w(s)^2 (1 - y^2) \rho + \frac{p+1}{p-1} \int_{-1}^{1} w(s)^2 \rho + \frac{1}{2} \int_{-1}^{1} \partial_s w(s)^2 \rho \to \frac{p+1}{p-1} E_\infty$ as $s \to \infty$.

(iii) $\frac{1}{p+1} \int_{-1}^{1} |w(y, s)|^{p+1} \rho \to \frac{2}{p-1} E_\infty$ as $s \to \infty$.

**Remark.** Part (i) of this proposition gives a different proof of the result of [18] when $k = 1$. However, the dependence of the bound on initial data is less clear here. Note that in the characteristic case, our new estimate is stronger than that of [18]. In addition, the energy partition we obtain in (ii) and (iii) is the same as for a stationary solution (see (49)).

Let us first establish two preliminary lemmas.

**Lemma 3.6.** There is a $C_0 > 0$ such that for all $s \geq s_0 + \frac{3}{2}$,
\[
\int_{-1}^{1} \frac{w(y, s)^2}{1 - y^2} \rho(y) dy \leq C_0.
\]
Proof. From the Hardy–Sobolev estimate (21) and (E4), we obtain

\[ \forall s \geq s_0 + \frac{3}{2}, \quad \int_{s - \frac{1}{2}}^{s + \frac{1}{2}} \int_{-1}^{1} \frac{w(y, s')^2}{1 - y^2} \rho(y) \, ds' \, dy \leq C_0 \]  

(70)

for some \( C_0 > 0 \). Thus, there is \( s_1(s) \in [s - \frac{1}{2}, s] \) such that

\[ \int \frac{w(y, s_1)^2}{1 - y^2} \rho(y) \, dy \leq 2C_0. \]

We then have from (E3) and (70),

\[
\int_{-1}^{1} \frac{w(y, s)^2}{1 - y^2} \rho(y) \, dy \\
= \int_{-1}^{1} \frac{w(y, s_1)^2}{1 - y^2} \rho(y) \, dy + 2 \int_{s_1}^{s} \int_{-1}^{1} \frac{w \partial_y w(y, s')}{1 - y^2} \rho(y) \, dy \, ds',
\]

\[ \leq 2C_0 + \left( \int_{s_1}^{s} \int_{-1}^{1} \frac{w^2(y, s')}{1 - y^2} \rho(y) \, ds' + \int_{s_1}^{s} \frac{1}{s_1} \int_{-1}^{1} \frac{\partial_s w^2(y, s')}{1 - y^2} \rho(y) \, ds' \right) \leq C'_0
\]

and the conclusion of Lemma 3.6 follows.

We now have from the proof of (E4) given in [17] a refinement of the estimates:

Lemma 3.7. There are \( s_1(s) \) and \( s_2(s) \) defined for \( s \geq s_0 + 1 \) such that:

(i) \[ \left| s_1(s) - s \right| + \left| s_2(s) - s \right| \to 0 \quad \text{as} \quad s \to \infty. \]

(ii) \[ \int_{s_1(s)}^{s_2(s) + 1} \int_{-1}^{1} \frac{|w(y, s)|^{p+1}}{p + 1} \rho \to \frac{2}{p - 1} E_\infty \quad \text{and} \]

\[
\int_{s_1(s)}^{s_2(s) + 1} \int_{-1}^{1} \left\{ \frac{1}{2} \partial_y w(y, s)^2 (1 - y^2) \rho + \frac{1}{2} \partial_s w(y, s)^2 \right\} \to \frac{p + 1}{p - 1} E_\infty
\]

as \( s \to \infty. \)
**Proof.** Remark from [17, identity (11), p. 1152] that we have for all \( s_1 \geq s_0 \) and \( s_2 \geq s_0 + 1 \),

\[
\frac{p-1}{2(p+1)} \int_{s_1}^{s_2+1} \int_{-1}^{1} |w(y,s)|^{p+1} \rho \, ds \, dw
\]

\[
= \int_{s_1}^{s_2+1} E(w(s)) \, ds + \frac{1}{2} \left[ \int_{-1}^{1} w \partial_y w \rho \right]_{s_1}^{s_2+1} \]

\[
+ \int_{s_1}^{s_2+1} \int_{-1}^{1} \left\{ -\partial_y w(y,s)^2 \rho - \partial_y wy \partial_y \rho - \partial_y wy \partial_y \rho + \frac{5-p}{2(p-1)} w \partial_y w \rho \right\} .
\]

Then, using (E3), we claim that for \( s \geq s_0 + 1 \), there are \( s_1(s) \) and \( s_2(s) \) such that Lemma 3.7(i) holds,

\[
\int_{s_1(s)}^{s_2(s)} \left| \int_{-1}^{1} \partial_y w(s') \right|^{p-1} ds' \to 0 \quad \text{as} \quad s \to \infty.
\]

Indeed, if \( \eta(s) = \int_{s}^{s_2(s)} \left( \int_{-1}^{1} \partial_y w(s') \right)^2 \frac{\rho}{1-y^2} \, ds' \), then (E3) implies that \( \eta(s) \to 0 \) as \( s \to \infty \). Therefore, considering \( s_1(s) \in [s, s + \sqrt{\eta(s)}] \) such that

\[
\int_{-1}^{1} \left( \int_{s_1(s)}^{s_2(s)} \partial_y w(s') \right)^2 \frac{\rho}{1-y^2} \, ds' \to 0,
\]

we conclude for \( s_1(s) \). Taking \( s_2(s) = s_1(s + 1) - 1 \) closes the proof.

Now, using (E2), (E3) and (E4), we see that \( \left[ \int_{-1}^{1} w \partial_y w \rho \right]_{s_1}^{s_2+1} \to 0 \) and

\[
\int_{s_1}^{s_2+1} \int_{-1}^{1} \left\{ -\partial_y w(y,s)^2 \rho - \partial_y wy \partial_y \rho - \partial_y wy \partial_y \rho + \frac{5-p}{2(p-1)} w \partial_y w \rho \right\} \to 0
\]
as \( s \to \infty \).

Since \( E(w(s)) \to E_{\infty} \) and \( |s_1(s) - s| + |s_2(s) - s| \to 0 \) as \( s \to \infty \), we get

\[
\int_{s_1}^{s_2+1} E(w(s)) \, ds \to E_{\infty},
\]

and the conclusion follows for \( \int_{s_1}^{s_2+1} |w(y,s)|^{p+1} \rho \). Using the definition of \( E(w(s)) (15) \), we conclude the proof of Lemma 3.7.

Let us prove Proposition 3.5 now.
Proof of Proposition 3.5. We proceed by a priori estimates. Using $E(w(s))$, it is enough to prove that $\int_{-1}^{1} \left( \frac{1}{2} \partial_y w(y, s)^2 (1 - y^2) + \frac{1}{2} \partial_x w(y, s)^2 + \frac{p+1}{(p-1)^2} w(y, s)^2 \rho(y) dy \right)$ converges to $\frac{p+1}{p-1} E_\infty$ as $s \to \infty$.

We have from Lemma 3.7 and (E3) that for all $\epsilon_0 \in (0, \frac{p+1}{(p-1)^2})$, there is $s_\epsilon \geq s_0 + 5$ such that for all $s \geq s_\epsilon$, we have:

$$\int_{s_1(s-2)}^{s_2(s-2)+1} \left| \int_{-1}^{1} \left( \frac{1}{2} \partial_y w(y, s')^2 (1 - y^2) + \frac{1}{2} \partial_x w(y, s')^2 + \frac{p+1}{(p-1)^2} w(y, s')^2 \rho(y) ds' - \frac{p+1}{p-1} E_\infty \right) \right| \leq \frac{\epsilon_0}{2}, \quad (71)$$

$$\int_{s_2(s-2)-1}^{s} \int_{s_1(s-2)}^{1} \frac{\partial x w(y, s')^2}{1 - y^2} \rho ds' dy \leq \delta_0(\epsilon_0), \quad (72)$$

and $|s_1(s) - s| + |s_2(s) - s| \leq \delta_0(\epsilon_0)$, where small $\delta_0$ will be fixed later dependent on $\epsilon_0$.

We now claim for all $s \geq s_\epsilon$,

$$\int_{-1}^{1} \left| \frac{1}{2} \partial_y w(y, s)^2 (1 - y^2) \rho + \frac{1}{2} \partial_x w(y, s)^2 \rho + \frac{p+1}{(p-1)^2} w(y, s)^2 \rho - \frac{p+1}{p-1} E_\infty \right| \leq \epsilon_0, \quad (73)$$

which concludes the proof of Proposition 3.5.

Proof of (73). From (71), we know that for all $s \geq s_\epsilon$, there is $s_3(s) \in [s_1(s-2), s_2(s-2)+1]$ such that,

$$\left| (1 + s_2 - s_1) \int_{-1}^{1} \left[ \frac{\partial_y w(s_3)^2}{2} (1 - y^2) + \frac{\partial_x w(s_3)^2}{2} + \frac{p+1}{(p-1)^2} w(s_3)^2 \right] \rho - \frac{p+1}{p-1} E_\infty \right| \leq \frac{\epsilon_0}{2},$$

therefore,

$$\left| \int_{-1}^{1} \left[ \frac{\partial_y w(s_3)^2}{2} (1 - y^2) + \frac{\partial_x w(s_3)^2}{2} + \frac{p+1}{(p-1)^2} w(s_3)^2 \right] \rho - \frac{p+1}{p-1} E_\infty \right| \leq \frac{\epsilon_0}{2} + C_0 \delta_0, \quad (74)$$

where $s_3 \in [s - 3, s - \frac{1}{2}]$.

If we impose that $C_0 \delta_0 < \frac{\epsilon_0}{2}$, then (73) holds for $s_3$. Let us prove (73) for all $s' \in [s_3, s]$ if $\epsilon_0$ is small enough and $\delta_0$ is small enough in terms of $\epsilon_0$.

By contradiction, assume that (73) holds for all $s' \in [s_3, s_4]$ and that for $s' = s_4$, we have equality in (73), where $s_4 \in [s_3, s]$. Then, from (23) and (73), we have for all $s' \in [s_3, s_4]$, $\|w(s')(1 - y^2)^{\frac{p+1}{p-1}}\|_{L^\infty} \leq C_0 (E_\infty + 1)$. Thus, using the derivative of the Lyapunov functional (16) and Lemma 3.6, we have for all $s' \in [s_3, s_4]$,
Integrating in time between $s_3$ and $s_4$, we obtain from (74) and (72),

$$
\epsilon_0 \leq \frac{\epsilon_0}{2} + C_0 \delta_0 + C_0 \delta_0 + C_0 \left( \int_{s_3}^{s_4} \left( \int_{-1}^{1} \frac{\partial_s w^2}{1 - y^2} \rho \right)^{1/2} \right) \leq \frac{\epsilon_0}{2} + C_0 (\delta_0 + \delta_0^{1/2}).
$$

Therefore, we obtain a contradiction by taking $\delta_0 = \epsilon_0^4$ and $\epsilon_0$ small enough. Thus, (73) is proved. This concludes the proof of Proposition 3.5. □

Note in addition that from Proposition 3.5 and (23), there is $C_0 > 0$ such that

$$
\forall s \geq s_0 + 3 \text{ and } y \in (-1, 1), \quad |w(y, s)(1 - y^2)^{1/p-1}| \leq C_0. \quad (75)
$$

From the dispersion property of the flow (16), we are able to prove that any recurrent nonlinear object in the dynamics as $s \to \infty$ is a stationary solution. Considering the space variable $\xi$ which allows us to write easily decoupling properties:

$$
\xi = \frac{1}{2} \log \left( \frac{1 + y}{1 - y} \right) \in \mathbb{R} \quad \text{(i.e. } y = \tanh \xi) \quad \text{and} \quad \bar{w}(\xi, s) = w(y, s)(1 - y^2)^{1/p-1}, \quad (76)
$$

we have the following proposition.

**Proposition 3.8** (Local convergence under a non-vanishing condition). Consider a sequence $(y_n, s_n)$ and $\epsilon_0 > 0$ such that $s_n \to \infty$ and $|w(y_n, s_n)(1 - y_n^2)^{1/p-1}| \geq \epsilon_0$. Then, there is $\xi_0 \in \mathbb{R}$ and $\omega_0 = \pm 1$ such that up to a subsequence we have:

(i) \[ \bar{w}(\xi + \xi_n, s + s_n) - \omega_0 \frac{\kappa_0}{\cosh^{p-1}(\xi - \xi_0)} \to 0 \] as $n \to \infty$, uniformly on compact sets of $|\xi| + |s|$ where $\xi_n = \frac{1}{2} \log \left( \frac{1 + y_n}{1 - y_n} \right)$.

(ii) \[ \forall M > 0, \quad \int_{\{y||\xi - \xi_n| < M\}} |w(y, s_n) - \omega_0 \kappa(d_n, y)|^{p+1} \rho \, dy \to 0 \] (78)
as $n \to \infty$, where
\[ d_n = \tanh \tilde{\xi}_n \quad \text{and} \quad \tilde{\xi}_n \text{ is such that } \xi_n + \tilde{\xi}_n = -\xi_0. \] (79)

**Remark.** We have for all $n \in \mathbb{N}$, $1/C \leq \frac{1-y_2}{1-d_n^2} \leq C$.

**Proof of Proposition 3.8.** Arguing as for (24) and (25), we see from (76), Proposition 3.5, Lemma 3.6 and (E3), that there is $C_0 > 0$ such that
\[ \forall s \geq s_0 + 3, \quad \|\bar{\psi}\|_{H^1(\mathbb{R})} \leq C_0, \] (80)
\[ \int_s^\infty \int_\mathbb{R} \partial_s \bar{\psi}^2 \, ds \, d\xi \leq C_0. \] (81)

Recall from (52) that the corresponding set of stationary solutions in $H^1(\mathbb{R})$ in the $\bar{\psi}$ variable (to the stationary solution in $H^0$ in the $\psi$ variable) is
\[ \pm \frac{\kappa_0}{\cosh \frac{2}{p-1} (\xi - \xi_0)}, \quad \text{where } \xi_0 \in \mathbb{R}. \] (82)

Proposition 3.8 reduces then to prove that up to a subsequence (also denoted by $s_n$) and for some $\bar{\psi}^* \not\equiv 0$, a stationary solution (that is a solution of Eq. (51)), we have
\[ \left| \bar{\psi}(\xi + \tilde{\xi}_n, s + s_n) - \bar{\psi}^*(\xi) \right| \to 0 \quad \text{as } n \to \infty \] (83)
uniformly on compact sets $|\xi| + |s| \leq M$.

Indeed, if (83) holds, then (i) of Proposition 3.8 follows from the fact that a non-zero stationary solution $\bar{\psi}^*$ is given by (82).

As for (ii) of Proposition 3.8, remark from (75) and (79) that
\[
\begin{aligned}
&\int_{\{y|\xi - \tilde{\xi}_n| \leq M\}} \left| w(y, s_n) - \omega_0 \kappa (d_n, y) \right|^p \rho \, dy \\
&\leq C_0 \int_{\{|y| \leq M\}} \left| w(y, s_n)(1 - y^2)^{\frac{1}{p-1}} - \omega_0 \kappa (d_n, y)(1 - y^2)^{\frac{1}{p-1}} \right|^2 \, dy \\
&\leq C_0 \int_{|\xi - \tilde{\xi}_n| \leq M} \left| \bar{\psi}(\xi, s_n) - \omega_0 \frac{\kappa_0}{\cosh \frac{2}{p-1} (\xi + \tilde{\xi}_n)} \right|^2 \, d\xi \\
&\leq C_0 \int_{|\xi| \leq M} \left| \bar{\psi}(\xi_n + \xi, s_n) - \omega_0 \frac{\kappa_0}{\cosh \frac{2}{p-1} (\xi - \xi_0)} \right|^2 \, d\xi \to 0
\end{aligned}
\] as $n \to \infty$ using (77). Thus, we just need to prove (83).
Proof of (83). The proof is similar to the one of Lemma 3.3. From (80), there is a subsequence $s_n$ and $\bar{w}^* \in H^1(\mathbb{R})$ such that
\[ \bar{w}(\xi + \xi_n, s_n) \rightarrow \bar{w}^*(\xi) \in C(\{\xi| < M\}) \text{ for all } M > 0. \] (84)

Remark from (76) and the hypotheses of Preposition 3.8 that
\[ |\bar{w}(\xi_n, s_n)| \geq \epsilon_0, \quad \text{thus } |\bar{w}^*(0)| \geq \epsilon_0 \quad \text{and } \bar{w}^* \not\equiv 0. \] (85)

Moreover, (80) and (81) give for all $M > 0$ and $|s| < M$,
\[
\int_{|\xi| < M} \left| \bar{w}(\xi_n + \xi, s_n + s) - \bar{w}^*(\xi) \right|^2 d\xi \\
\leq \int_{|\xi| < M} |\bar{w}(\xi_n + \xi, s_n) - \bar{w}^*(\xi)|^2 d\xi + C_0 \int_{s_n - M}^{s_n + M} \left( \int_{|\xi| < M} |\partial_s \bar{w}(\xi_n + \xi, s')|^2 d\xi \right)^{1/2} ds' \\
\leq \int_{|\xi| < M} |\bar{w}(\xi_n + \xi, s_n) - \bar{w}^*(\xi)|^2 d\xi + C_0 \sqrt{M} \left( \int_{s_n - M}^{s_n + M} \int_{|\xi'| < M} |\partial_s \bar{w}(\xi, s')|^2 d\xi ds' \right)^{1/2} \rightarrow 0
\]
as $n \rightarrow \infty$, and from the fact that
\[ \|\bar{w}\|_{L^\infty(\{\xi| < M\})} \leq C_0\|\bar{w}\|_{L^2(\{\xi| < M + 1\})}\|\bar{w}\|_{H^1(\{\xi| < M + 1\})}, \]
we have (83) with $\bar{w}^*$ defined in (84). It remains to prove that $\bar{w}^*(\xi)$ corresponds to a stationary solution. Let us remark from similar computations to p. 59 that
\[ (1 - y^2)^{\frac{1}{p+1}} \left[ -\partial_{ss}^2 w - \frac{p+3}{p-1} \partial_s w - 2y\partial_{ss}^3 w + Lw - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w \right] \\
= - \frac{\partial_{ss}^2 \bar{w}}{\cosh^2 \xi} + \left( \frac{\tanh^2 \xi - \frac{p+3}{p-1}}{p-1} \right) \partial_s \bar{w} - 2 \tanh \xi \partial_{ss} \bar{w} + \bar{w} \xi \bar{w} - \frac{4}{(p-1)^2} \bar{w} + |\bar{w}|^{p-1} \bar{w}
\]
and thus for all $\bar{\varphi}(\xi, s) C^\infty$ with compact support included in $\{s \geq s^*\}$,
\[
\int_{\mathbb{R}} \left( \frac{1}{\cosh^2 \xi} \partial_s \bar{\varphi} d\xi ds - \left( \frac{p+3}{p-1} - \tanh^2 \xi \right) \bar{\varphi} + \partial_s (2\bar{\varphi} \tanh^2 \xi) \right) \partial_s \bar{w} d\xi ds \\
+ \int_{\mathbb{R}} \left( \bar{w} \xi \bar{\varphi} - \frac{4}{(p-1)^2} \bar{w} \bar{\varphi} + |\bar{w}|^{p-1} \bar{w} \bar{\varphi} \right) d\xi ds \\
= \int_{\mathbb{R}} \left( \partial_s \varphi - \frac{p+3}{p-1} \varphi + \frac{1}{\rho} \partial_s (2\rho \varphi) \right) \partial_s w \rho dy ds \\
+ \int_{\mathbb{R}} \left( L(\varphi) w - \frac{2(p+1)}{(p-1)^2} w \varphi + |w|^{p-1} w \varphi \right) \varphi \rho dy ds = 0
\] (86)
with $\varphi(y, s) = \bar{\varphi}(\xi, s)$. The fact that the latter expression is zero follows from the same computations to the non-characteristic case (see Step 1 in Section 3.1).

Consider now an arbitrary $\bar{\varphi}_1(\xi) \in C^\infty$ compactly supported. Apply identity (86) with

$$\bar{\varphi}(\xi, s) = \bar{\varphi}_1(\xi - \xi_n) \bar{\varphi}_2(s - s_n)$$

where $\bar{\varphi}_2 \in C^\infty_c$, supp $\bar{\varphi}_2 \in [-2, 2]$ and $\int \bar{\varphi}_2 = 1$. Since we know from (81) that $\int_{s_n - 2}^{s_n + 2} \int \partial_x \bar{w}^2 \to \infty$ as $n \to \infty$, we use (83) and the Cauchy–Schwarz inequality to get as $n \to \infty$,

$$\int \bar{w}^* \partial_{\xi \xi} \bar{\varphi}_1 + \int \left( |\bar{w}^*|^{p-1} \bar{w}^* - \frac{4}{(p-1)^2} \bar{w}^* \right) \bar{\varphi}_1 = 0.$$ 

From the fact that $\bar{\varphi}^* \in H^1(\mathbb{R})$ and classical elliptic regularity theory, we have $\bar{\varphi}^* \in C^2(\mathbb{R})$ and $\bar{\varphi}^*$ satisfies

$$\partial_{\xi \xi} \bar{\varphi}^* + |\bar{\varphi}^*|^{p-1} \bar{\varphi}^* - \frac{4}{(p-1)^2} \bar{\varphi}^* = 0 \quad \text{for } \xi \in \mathbb{R},$$

which concludes the proof of Proposition 3.8. \qed

**Part 2.** Conclusion of the proof of Theorem 2 in the characteristic case. From (E1), we know that $E_\infty \geq 0$. If $E_\infty = 0$, then from Proposition 3.5, we have $\|w(s)\|_{H^4} \to 0$ as $s \to \infty$ and the conclusion is valid with $k = 0$. Assume from now on that $E_\infty > 0$.

**Step 1. Localization of the energy packets.** Remark first from Proposition 3.5, Lemma 3.6 and (23), that there is $C_0 > 0$ and $s_1 \geq s_0 + 3$ such that for all $s \geq s_1$,

$$\int_{-1}^{1} \frac{w(s)^2}{1 - y^2} \rho + \int_{-1}^{1} \frac{\partial_x w(y, s)^2}{1 - y^2} \rho + \|w(s)(1 - y^2)^{\frac{1}{p-1}}\|_{L^\infty} \leq C_0 \quad \text{and} \quad \int_{-1}^{1} |w(s)|^{p+1} \rho \geq \frac{1}{C_0}.$$ 

Therefore,

$$\frac{1}{C_0} \leq \int |w|^{p+1} \rho \leq \int \frac{w^2}{1 - y^2} \rho \|w(1 - y^2)^{\frac{1}{p-1}}\|_{L^\infty}^{p-1} \leq C_0 \|w(1 - y^2)^{\frac{1}{p-1}}\|_{L^\infty}^{p-1},$$

hence, there exists $\epsilon_0 \in (0, \frac{\kappa_0}{4})$ such that for all $s \geq s_1$,

$$\|w(s)(1 - y^2)^{\frac{1}{p-1}}\|_{L^\infty}^{p-1} \geq 2\epsilon_0. \quad (87)$$

In particular, if we define

$$\tilde{A}(s) = \{ \xi \mid |\bar{w}(\xi, s)| \geq \epsilon_0 \} \quad \text{and} \quad A(s) = \{ \xi \mid d(\xi, \tilde{A}(s)) < 1 \},$$

then, for all $s \geq s_1^*$, $\tilde{A}(s) \neq \emptyset$ and $A(s) \neq \emptyset$. We now have the following lemma.
Lemma 3.9. There is $k \in \mathbb{N}^*$, $s_2$ and $\mu_0 > 0$ such that for all $s \geq s_2$,

(i) $A(s) = \bigcup_{i=1}^k (\xi_i(s) - \mu_i(s), \xi_i(s) + \mu_i(s))$ where $\xi_i(s)$ is a continuous function of $s$,

$$|\xi_i(s) - \xi_j(s)| \to \infty \quad \text{for} \quad i \neq j \quad \text{and} \quad \mu_i(s) \to \mu_0$$

as $s \to \infty$.

(ii) $|\tilde{w}(\xi + \xi_i(s), s) - \omega_i \frac{k_0}{\cosh \frac{p}{2} (\xi)}| \to 0$

uniformly on compact sets of $|\xi|$, where $\omega_i = \pm 1$.

(iii) For all $\epsilon > 0$, there exist $M_\epsilon > 0$ and $s_\epsilon \geq s_2$ such that if $s \geq s_\epsilon$ and $\inf_{i=1, \ldots, k} |\xi - \xi_i(s)| > M_\epsilon$, then $|\tilde{w}(\xi, s)| \leq \epsilon$.

Proof. (i), (ii) Note first that $A(s)$ is an open set of $\mathbb{R}$, that is a disjoint union of open intervals. Let $k(s) \in \mathbb{N}$ be the number of connected components of $A(s)$. Let us show that for $s$ large enough,

$$k(s) \leq 2 \frac{E_\infty}{E(\kappa_0)} + 1.$$  \hspace{1cm} (89)

Let us assume by contradiction that for some $m > 2 \frac{E_\infty}{E(\kappa_0)} + 1$, there are $s_n \to \infty$, $\xi_{1,n} < \cdots < \xi_{m,n}$ in $\mathbb{R}$, and positive $\mu_{1,n}, \ldots, \mu_{k,n}$ such that $(\xi_{i,n} - \mu_{i,n}, \xi_{i,n} + \mu_{i,n})$ are disjoint and $A(s_n) \supset \bigcup_{i=1}^k (\xi_{i,n} - \mu_{i,n}, \xi_{i,n} + \mu_{i,n})$. By definition of $A(s_n)$, there exist $\xi'_{i,n} \in (\xi_{i,n} - 1, \xi_{i,n} + 1) \cap \tilde{A}(s_n)$ such that $|\tilde{w}(\xi'_{i,n}, s_n)| \geq \epsilon_0$. Therefore, it follows from Proposition 3.5 that up to a subsequence and for all $i = 1, \ldots, m$,

$$|\tilde{w}(\xi + \xi_{i,n}, s + s) - \omega_i \frac{k_0}{\cosh \frac{p}{2} (\xi - \xi_i)}| \to 0 \quad \text{uniformly for} \quad |\xi| + |s| \leq M$$  \hspace{1cm} (90)

for some $x_i \in \mathbb{R}$ and $\omega_i = \pm 1$. Moreover, since $(\xi_{i,n} - \mu_{i,n}, \xi_{i,n} + \mu_{i,n})$ is a connected component of $A(s)$ with center $\xi_{i,n}$, we use (90) and the fact that

$$\frac{k_0}{\cosh \frac{p}{2} (\xi)} > \epsilon_0 \quad \text{iff} \quad -\mu'_0 \leq \xi \leq \mu'_0 \quad \text{for some} \quad \mu'_0 = \mu'_0(\epsilon_0) > 0$$

to derive that for all $i = 1, \ldots, m$:

- $x_i = 0$,
- $\mu_{i,n} \to \mu'_0 + 1$ (use the fact that for any $\delta > 0$ and $n$ large enough, we have $\tilde{A}(s_n) \cap (\xi_{i,n} - 2(\mu'_0 + 1), \xi_{i,n} + 2(\mu'_0 + 1)) \subset (\xi_{i,n} - (\mu'_0 + \delta), \xi_{i,n} + (\mu'_0 + \delta))$,
- $|\xi_{i,n} - \xi_{j,n}| \to \infty$ as $n \to \infty$, for $i \neq j$.

Making the change of variables $y = \tanh \xi$, we see from (49) that

$$\int_{\mathbb{R}} \left| \frac{k_0}{\cosh \frac{p}{2} (\xi)} \right|^{p+1} d\xi = k_0^{p+1} \int_{-1}^1 \rho(y) dy = \frac{2(p+1)}{p-1} E(k_0).$$
Fix then $M > 0$ such that
\[ \int_{|\xi| > M} \left| \frac{\kappa_0}{\cosh^{p-1}(\xi)} \right|^{p+1} d\xi < \frac{2}{100} \frac{2}{p-1} E(\kappa_0), \]
and, by Proposition 3.5 and (90), take $n \geq n_0(M)$ such that the intervals $(\xi_{i,n} - M, \xi_{i,n} + M)$ are disjoint for $i = 1, \ldots, m$ and
\[ \frac{2(p + 1)}{p - 1} \left( E_\infty + \frac{1}{100} E(\kappa_0) \right) \]
\[ \geq \int |w(y,s_n)|^{p+1} \rho \, dy = \int |\bar{w}(\xi, s_n)|^{p+1} d\xi \geq \sum_{i=1}^{m} \int |\bar{w}(\xi, s_n)|^{p+1} d\xi \]
\[ \geq \sum_{i=1}^{m} \left( \int_{\mathbb{R}} \left| \frac{\kappa_0}{\cosh^{p-1}(\xi)} \right|^{p+1} d\xi - \frac{2}{100} \frac{2(p + 1)}{p - 1} E(\kappa_0) \right) = m \frac{2(p + 1)}{p - 1} E(\kappa_0) \left( 1 - \frac{2}{100} \right), \]

hence, $m \leq \frac{100 \cdot E_\infty}{E(\kappa_0)} + \frac{1}{98}$, which is a contradiction. Thus, (89) holds.

Let us show now that $k(s)$ is constant for $s$ large, that is, $k(s) = k \in \mathbb{N}^*$. We proceed by contradiction and consider $s_n \to \infty$ and $\delta_n \in (-1, 1)$ such that $k(s_n + \delta_n) < k(s_n) = m$. Making the same construction for $s_n$ as we did for the previous proof, defining in particular $\xi_{1,n} < \cdots < \xi_{m,n}$ in $\mathbb{R}$, we see that (90) holds with $x_i = 0$. Applying (90) with $s = \delta_n \in (-1, 1)$, we see that $A(s_n + \delta_n)$ has at least $m$ connected components inherited from those of $A(s_n)$ (here we use the fact that $\epsilon_0 < \frac{\kappa_0}{2}$). Contradiction. Thus, $k(s) = k \in \mathbb{N}^*$ for $s \geq s_2$ for some $s_2$ large enough.

We are now able to define for all $s \geq s_2$, $\xi_1(s) < \cdots < \xi_k(s)$, $\mu_i(s)$ such that (88) holds. Note that (90) writes
\[ \left| \bar{w}(\xi + \xi_i(s), s + \sigma) - \omega_i(s) \frac{\kappa_0}{\cosh^{p-1}(\xi)} \right| \to 0 \quad \text{as } s \to \infty \text{ uniformly for } |\xi| + |\sigma| \leq M, \]
for some $\omega_i(s) = \pm 1$. In particular, $\xi_i(s)$ is a continuous function of $s$ and $\omega_i(s)$ is a constant for $s$ large. This concludes the proof of (i) and (ii) of Lemma 3.9.

(iii) This estimate follows by contradiction considering some $\epsilon_1 \in (0, \frac{\kappa_0}{2})$ and $(\xi_n, s_n)$ such that $s_n \to \infty$, $\min_{i=1, \ldots, k} |\xi_n - \xi_i(s_n)| \to \infty$ and $|\bar{w}(\xi_n, s_n)| \geq \epsilon_1$. Applying Proposition 3.8 and the fact that $\epsilon_1 \leq \frac{\kappa_0}{4}$, we see that $\text{dist}(\xi_n, A(s_n)) \leq M_1(\epsilon_1)$, which is a contradiction. This concludes the proof of Lemma 3.9. \(\Box\)

Using the fact that
\[ |\xi_i(s) - \xi_j(s)| \to \infty \quad \text{as } s \to \infty \text{ for } i \neq j, \quad (91) \]
we have the following.

Claim 3.10. If
\[ d_i(s) = -\tanh \xi_i(s), \quad (92) \]
then we have as \( s \to \infty \):

\[
\int_{-1}^{1} \left| \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right|^{p+1} \rho - \left( \sum_{i=1}^{k} \int_{-1}^{1} \kappa(d_i(s), y)^{p+1} \right) \to 0,
\]

\[
\int_{-1}^{1} \left( \sum_{i=1}^{k} \omega_i \partial_y \kappa(d_i(s), y) \right)^2 (1 - y^2) \rho - \left( \sum_{i=1}^{k} \int_{-1}^{1} (\partial_y \kappa(d_i(s), y))^2 (1 - y^2) \right) \to 0,
\]

\[
\int_{-1}^{1} \left( \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right)^2 \rho - \left( \sum_{i=1}^{k} \int_{-1}^{1} \kappa(d_i(s), y)^2 \rho \right) \to 0,
\]

\[
\int_{-1}^{1} \left( \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right) \left( \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)^p \right) \rho - \int_{-1}^{1} \sum_{i=1}^{k} \kappa(d_i(s), y)^{p+1} \rho \to 0.
\]

**Proof.** We only prove the first inequality since the two others follow in the same way. Since \( \kappa(d_i(s), y) \) becomes \( \kappa_0 / \cosh^{\frac{2}{p+1}} (\xi - \xi_i(s)) \) by the transformation (76), we use the linear character of (76) to get

\[
\int_{-1}^{1} \left| \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right|^{p+1} \rho = \int_{\mathbb{R}} \left| \sum_{i=1}^{k} \omega_i \kappa_0 \cosh^{\frac{2}{p+1}} (\xi - \xi_i(s)) \right|^{p+1} d\xi.
\]

Since we know from (91) that

\[
\int_{\mathbb{R}} \left( \left| \sum_{i=1}^{k} \omega_i \kappa_0 \cosh^{\frac{2}{p+1}} (\xi - \xi_i(s)) \right|^{p+1} - \sum_{i=1}^{k} \left| \kappa_0 \cosh^{\frac{2}{p+1}} (\xi - \xi_i(s)) \right|^{p+1} \right) d\xi \to 0
\]

as \( s \to \infty \), we just use again (76) to conclude the proof of Claim 3.10. \( \square \)

**Step 2. Conclusion of the proof.** We want to prove that

\[
q(y, s) \equiv w(y, s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \to 0
\]

in the energy norm. Using Step 1, we first prove the convergence in \( L_\rho^{p+1} \). From (iii) in Proposition 3.5, this implies the quantization of \( E_\infty \). Then, using the weak convergence of \( q(s) \) to 0 in the energy space and the convergence of the norm in (ii) of Proposition 3.5, we prove the strong convergence.

Let us prove now the following.
Claim 3.11 (Convergence in $L^{p+1}_\rho$). As $s \to \infty$,

$$\int \left| w(s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right|^{p+1} \rho \to 0 \quad \text{and} \quad \int \left| w(s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right|^2 \rho \to 0. \quad (93)$$

**Proof.** Remark first that the Hölder inequality and the $L^{p+1}$ estimate imply the $L^2$ estimate. Let us then prove the $L^{p+1}$ estimate.

For all $\epsilon > 0$, there are from (iii) of Lemma 3.9 $M_\epsilon > 0$ and $s_\epsilon$ such that if $s \geq s_\epsilon$ and $\forall i = 1, \ldots, k, |\xi - \xi_i(s)| \geq M_\epsilon$, then

$$|w(y, s)| \left(1 - y^2\right)^{\frac{1}{p+1}} \leq \frac{\epsilon}{2},$$

$$\left| \sum_{i=1}^{k} \kappa(d_i(s), y) \right| \left(1 - y^2\right)^{\frac{1}{p+1}} = \sum_{i=1}^{k} \frac{\kappa_0}{\cosh^{\frac{2}{p+1}}(\xi - \xi_i(s))} \leq \frac{\epsilon}{2},$$

$$\left| w(y, s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right| \left(1 - y^2\right)^{\frac{1}{p+1}} \leq \epsilon, \quad (94)$$

where $y = \tanh \xi$. Therefore, for $s \geq s_\epsilon$,

$$\int \left| w(s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s)) \right|^{p+1} \rho \leq \int \left| w(y, s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right|^{p+1} \rho$$

$$+ \sum_{i=1}^{k} \int \left| w(y, s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right|^{p+1} \rho,$$

$$\leq \epsilon^{p-1} \int \frac{|w(y, s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)|^2}{1 - y^2} \rho + o(1)$$

( from (94) and the fact that $|\xi_i(s) - \xi_j(s)| \to \infty$ as $s \to \infty$ for $i \neq j$). Therefore, for $s$ large,

$$\int \left| w(s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s)) \right|^{p+1} \rho \leq C_0 \epsilon^{p-1} \left( \|w(s)\|_{H^0}^2 + \sum_{i=1}^{k} \|\kappa(d_i(s))\|_{H^0}^2 \right) + o(1)$$

$$\leq C_0 \epsilon^{p-1} + o(1) \leq 2C_0 \epsilon^{p-1}$$

(from (23) and (ii) in Lemma 3.9). Letting $\epsilon \to 0$ allows us to conclude. \qed
As a consequence, we have the following energy constraint.

**Corollary 3.12 (Quantization of the limit of \( E(w(s)) \)).** It holds that \( E_\infty = k E(\kappa_0) \), where \( k \in \mathbb{N}^* \) was introduced in Lemma 3.9.

Indeed, on one hand, we have from Proposition 3.5

\[
\int_{-1}^{1} |w(s)|^{p+1} \rho \to \frac{2(p+1)}{p-1} E_\infty \quad \text{as } s \to \infty.
\]

On the other hand, from Claims 3.11 and 3.10, and (49), we have

\[
\lim_{s \to \infty} \int_{-1}^{1} |w(s)|^{p+1} \rho = \lim_{s \to \infty} \int_{-1}^{1} \left( \sum_{i=1}^{k} \kappa(d_i(s), y) \right)^{p+1} \rho = \lim_{s \to \infty} \sum_{i=1}^{k} \int_{-1}^{1} \kappa(d_i(s), y)^{p+1} \rho
\]

\[
= \lim_{s \to \infty} \sum_{i=1}^{k} 1 \int_{-1}^{1} \kappa_0^{|p+1|} \rho = k \int_{-1}^{1} \kappa_0^{|p+1|} \rho = \frac{2(p+1)}{p-1} k E(\kappa_0),
\]

and the corollary follows. \(\square\)

We now have the following.

**Claim 3.13.** If we define \( I(s) \) by

\[
I(s) = \frac{1}{2} \int_{-1}^{1} \left( \frac{1}{2} |\partial_y w - \sum_{i=1}^{k} \omega_i \partial_y \kappa(d_i(s))| \right)^2 (1 - y^2) + \frac{p+1}{(p-1)^2} \int_{-1}^{1} w - \sum_{i=1}^{k} \omega_i \kappa(d_i(s)) |^2 \left( \frac{1}{2} \left( \partial_y w \right)^2 \right) \rho,
\]

then we have \( I(s) \to 0 \) as \( s \to \infty \).

**Proof.** Note first that

\[
I(s) = \frac{1}{2} \int \partial_y w(y, s)^2 (1 - y^2) \rho + \frac{p+1}{(p-1)^2} \int w^2 \rho
\]

\[
+ \frac{1}{2} \int \partial_y w(y, s)^2 \rho + J(s) + K(s),
\]

where

\[
J(s) = \frac{1}{2} \int \left( \sum_{i=1}^{k} \omega_i \partial_y \kappa(d_i(s), y) \right)^2 (1 - y^2) \rho + \frac{(p+1)}{(p-1)^2} \int \left( \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right)^2 \rho,
\]

\[
K(s) = - \int \partial_y w \partial_y \left( \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right) (1 - y^2) \rho - \frac{2(p+1)}{(p-1)^2} \int w \left( \sum_{i=1}^{k} \omega_i \kappa(d_i(s)) \right) \rho.
\]
Using Claim 3.10 and (49), we see that

\[ J(s) = \sum_{i=1}^{k} \int \left( \frac{1}{2} \partial_y \kappa(d_i(s), y) \right)^2 (1 - y^2) \rho + \frac{(p + 1)}{(p - 1)^2} \left( \kappa(d_i(s), y) \right)^2 \rho + o(1) \]

\[ = k \frac{(p + 1)}{p - 1} E(\kappa_0) + o(1). \quad (96) \]

We claim that

\[ K(s) \to -2 \frac{k(p + 1)}{p - 1} E(\kappa_0) \text{ as } s \to \infty. \quad (97) \]

Indeed, from integration by parts and the fact that \( \kappa(d_i(s), \cdot) \) is a solution of (47), we have

\[ K(s) = \int w(s) \left[ \sum_{i=1}^{k} \left( \frac{1}{\rho} \partial_y (\omega_i \partial_y \kappa(d_i(s), y)) (1 - y^2) \rho \right) - \frac{2(p + 1)}{(p - 1)^2} \omega_i \kappa(d_i(s), y) \right] \rho \]

\[ = - \int w(s) \left[ \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)^p \right] \rho. \]

Therefore, from (49), Hölder’s inequality and Claims 3.11 and 3.10, we write

\[ K(s) = - \int \left[ \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \right] \left[ \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)^p \right] \rho + o(1) \]

\[ = - \sum_{i=1}^{k} \int \kappa(d_i(s), y)^{p+1} \rho + o(1) = -k \int \kappa_0^{p+1} \rho + o(1) = -2k \frac{(p + 1)}{p - 1} E(\kappa_0) + o(1), \]

which concludes the proof of (97).

Using (95), Proposition 3.5, (96) and (97), we write

\[ I(s) \to \frac{p + 1}{p - 1} E_\infty + k \frac{(p + 1)}{p - 1} E(\kappa_0) - 2k \frac{(p + 1)}{p - 1} E(\kappa_0) = \frac{p + 1}{p - 1} (E_\infty - k E(\kappa_0)) = 0 \]

by Claim 3.12, which proves Claim 3.13.

Claim 3.13 together with Corollary 3.12 conclude the proof of Theorem 2 in the characteristic case (use Lemma 3.9 and (92) for the continuity of \( d_i(s) \); use (88) and (92) to derive estimate (B.ii)).

4. The linearized operator around a non-zero stationary solution

In this section, we study the properties of the linearized operator of Eq. (7) around the stationary solution \( \kappa(d, y) \) (13).
If we introduce $q = (q_1, q_2) = \left( \frac{q_1}{q_2} \right)$ for all $s \in [s_0, \infty)$ by
\[
\left( \begin{array}{c}
w(y, s) \\
\partial_s w(y, s)
\end{array} \right) = \left( \begin{array}{c}
\kappa(d, y) \\
0
\end{array} \right) + \left( \begin{array}{c}
q_1(y, s) \\
q_2(y, s)
\end{array} \right),
\]
then we see from Eq. (7) that $q$ satisfies the following equation for all $s \geq s_0$ (for the proof in a more general case, see the proof of Proposition 5.1(ii) below):
\[
\frac{\partial}{\partial s} \left( \begin{array}{c}
q_1 \\
q_2
\end{array} \right) = L_d \left( \begin{array}{c}
q_1 \\
q_2
\end{array} \right) + \left( \begin{array}{c}
0 \\
f_d(q_1)
\end{array} \right),
\]
(99)
where
\[
L_d \left( \begin{array}{c}
q_1 \\
q_2
\end{array} \right) = \left( \begin{array}{c}
\mathcal{L}q_1 + \psi(d, y)q_1 - \frac{p+3}{p-1}q_2 - 2yq_2' \\
r_1 \left( - \mathcal{L}r_1 + r_1' + q_2 r_2 (1 - y^2) + q_2 r_2 \right)
\end{array} \right),
\]
\[
f_d(q_1) = |\kappa(d, \cdot) + q_1|^{p-1} (\kappa(d, \cdot) + q_1) - \kappa(d, \cdot)^p - p \kappa(d, \cdot)^{p-1} q_1,
\]
(100)
$\mathcal{L}$, $\psi(d, \cdot)$ and $\kappa(d, \cdot)$ are defined respectively in (8), (41) and (13). In this section, we study the linear operator $L_d$ in the energy space $\mathcal{H}$ defined in (9). Note from (9) that we have
\[
\|q\|_{\mathcal{H}} = \left[ \phi(q, q) \right]^{1/2} < +\infty,
\]
(101)
where the inner product $\phi$ is defined by
\[
\phi(q, r) = \phi \left( \left( \begin{array}{c}
q_1 \\
q_2
\end{array} \right), \left( \begin{array}{c}
r_1 \\
r_2
\end{array} \right) \right) = \int_{-1}^{1} (q_1 r_1 + q_2 r_2 (1 - y^2) + q_2 r_2) \rho \, dy.
\]
(102)
Using integration by parts and the definition of $\mathcal{L}$ (8), we have the following identity:
\[
\phi(q, r) = \int_{-1}^{1} (q_1 (-\mathcal{L}r_1 + r_1) + q_2 r_2) \rho \, dy.
\]
(103)
One of the major difficulties in the proof of the convergence in Theorem 3 comes from the fact that the linear operator $L_d$ is not self-adjoint. In particular, standard spectral theory does not apply. Nevertheless, using a modified version of Proposition 2.3, one can directly show that
\[
\lambda_n = 1 - n \quad \text{and} \quad \mu_n = -2 \frac{(p+1)}{p-1} - n, \quad n \in \mathbb{N},
\]
are eigenvalues of $L_d$ and that the corresponding eigenfunctions are polynomials of degree $n$ that span the whole space $\mathcal{H}$. Note that $L_d$ has one positive direction ($\lambda = 1$) and one null direction ($\lambda = 0$), and the rest of the spectrum is negative ($\lambda \leq -1$). Then, one can expand the solution $q$ according to the positive, null and negative part of the spectrum. The general strategy is to obtain properties of $L_d$ with the hope to extend them to the nonlinear equation (99). From the Hamiltonian structure of the original equation or the non-self-adjoint character of $L_d$, few examples
are known in the literature where this strategy works. Indeed, the problem we are looking to is related to the so called existence and asymptotic stability of blow-up profile in the energy space (for $L^2$ critical generalized KdV, see Martel and Merle [13] and for $L^2$ critical NLS equation, see Merle and Raphaël [14]). In this section:

- We first show that $\lambda = 1$ and $\lambda = 0$ are eigenvalues of $L_d$ and compute explicitly the corresponding eigenfunctions (Lemma 4.2).
- Then, we compute explicitly eigenfunctions of $L_d^*$ (the adjoint of $L_d$ with respect to the inner product $\phi$) for $\lambda = 1$ and $\lambda = 0$, which will give projections on the corresponding eigenspace of $L_d$.
- Finally, subtracting from the solution the projections on eigenspaces of $\lambda = 1$ and $\lambda = 0$, we obtain the projection on the negative part of the spectrum. However, to control that part, no spectral theory will be used, because of the weakness and the technical character of such an approach in the Hamiltonian context. Instead, we use a different approach based on the nonlinear equation (99) and its dispersive relation. For the similar results in the context of KdV and NLS equations see the references.

4.1. The conjugate operator $L_d^*$

In the following, we compute $L_d^*$.

Lemma 4.1 (The conjugate operator of $L_d$ with respect to the inner product $\phi$). For any $|d| < 1$, the operator $L_d^*$ conjugate of $L_d$ with respect to $\phi$ is given by

\[ L_d^* \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1}r_2 + 2y\frac{q_2'}{q_2} - \frac{8}{(p-1)(1-y^2)}r_2 \\ \frac{R_d(r_2)}{r_2} \end{pmatrix} \]  

(104)

for any $(r_1, r_2) \in (\mathcal{D}(\mathcal{L}))^2$, where $r = R_d(r_2)$ is the unique solution of

\[-\mathcal{L}r + r = \mathcal{L}r_2 + \psi(d, y)r_2.\]  

(105)

Remark. The domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ defined in (8) is the set of all $r \in L^2_\rho$ such that $\mathcal{L}r \in L^2_\rho$.

Proof of Lemma 4.1. By definition of $L_d^*$, we have for all $q = (q_1, q_2)$ and $r = (r_1, r_2)$ in $\mathcal{H}$,

\[ \phi(L_d(q), r) = \phi(q, L_d^*(r)). \]  

(106)

Using (100) and (103), we write for arbitrary $(q_1, q_2)$ and $(r_1, r_2)$ in $\mathcal{H}$,

\[ \phi(L_d(q), r) = \phi\left(\begin{pmatrix} q_2 \\ \mathcal{L}q_1 + \psi(d, y)q_1 - \frac{p+3}{p-1}q_2 - 2yq_2' \end{pmatrix}, \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\right) \]

\[ = \int_{-1}^1 \left( q_2(-\mathcal{L}r_1 + r_1) + \left(\mathcal{L}q_1 + \psi(d, y)q_1 - \frac{p+3}{p-1}q_2 - 2yq_2'\right)r_2 \right) \rho dy. \]

Integrating by parts, we write
\[-2 \int_{-1}^{1} yq_2' r_2 \rho \, dy = 2 \int_{-1}^{1} q_2 (r_2 \rho + yr'_2 \rho + yr_2 \rho') \, dy \]
\[= 2 \int_{-1}^{1} q_2 \left( r_2 \rho + yr'_2 \rho - yr_2 \rho \frac{4 y \rho}{(p - 1)(1 - y^2)} \right) \, dy \]
\[= \int_{-1}^{1} q_2 \left( 2 \frac{p + 3}{p - 1} r_2 + 2yr'_2 - \frac{8r_2}{(p - 1)(1 - y^2)} \right) \rho \, dy. \quad (107)\]

Therefore, since \( \mathcal{L} \) is self-adjoint, we get

\[\phi(L_d(q), r) = \int_{-1}^{1} q_1 (\mathcal{L} r_2 + \psi(d, y)r_2) \rho \]
\[+ \int_{-1}^{1} q_2 \left( -\mathcal{L} r_1 + r_1 + \frac{p + 3}{p - 1} r_2 + 2yr'_2 - \frac{8r_2}{(p - 1)(1 - y^2)} \right) \rho \, dy. \quad (108)\]

Now, we define \( R_d : L^2_\rho(-1, 1) \rightarrow L^2_\rho(-1, 1) \) by (105). Note that \( R_d \) is well defined, whenever \( r_2 \) and \( \mathcal{L}r_2 \) are in \( L^2_\rho \) (or \( r_2 \in \mathcal{D}(\mathcal{L}) \)), since \( \mathcal{H}_0 \) equipped with the inner product

\[\langle u, v \rangle_{\mathcal{H}_0} = \int_{-1}^{1} (u'(y)v'(y)(1 - y^2) + u(y)v(y)) \rho(y) \, dy \]
\[= \int_{-1}^{1} (-\mathcal{L}u(y) + u(y)) v(y) \rho(y) \, dy \quad (109)\]

is a Hilbert space. Using (105), (108) and (103), we see that

\[\phi(L_d(q), r) = \int_{-1}^{1} q_1 (-\mathcal{L} R_d(r_2) + R_d(r_2)) \rho \]
\[+ \int_{-1}^{1} q_2 \left( -\mathcal{L} r_1 + r_1 + \frac{p + 3}{p - 1} r_2 + 2yr'_2 - \frac{8r_2}{(p - 1)(1 - y^2)} \right) \rho \]
\[= \phi\left( \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \left( -\mathcal{L} r_1 + r_1 + \frac{p + 3}{p - 1} r_2 + 2yr'_2 - \frac{8r_2}{(p - 1)(1 - y^2)} \right) \right). \]

Using the characterization of \( L^*_d \) by (106), we get (104). This concludes the proof of Lemma 4.1. \( \square \)
4.2. Non-negative directions of $L_d$

Let us now find non-negative directions of $L_d$. We claim the following.

**Lemma 4.2** (Non-negative eigenvalues and eigenfunctions for $L_d$).

(i) For all $|d| < 1$, $\lambda = 1$ and $\lambda = 0$ are eigenvalues of the linear operator $L_d$ and the corresponding eigenfunctions are respectively

\[
F_1^d(y) = (1 - d^2)^{\frac{p}{p-1}} \left( \frac{(1 + dy)^{-\frac{2}{p-1} - 1}}{(1 + dy)^{\frac{2}{p-1} - 1}} \right)
\]

and

\[
F_0^d(y) = (1 - d^2)^{\frac{1}{p-1}} \left( \frac{\frac{y+d}{(1+dy)^{\frac{2}{p-1} + 1}}}{0} \right).
\]

(ii) Moreover, it holds for some $C_0 > 0$ and any $\lambda \in \{0, 1\}$ that

\[
\forall |d| < 1, \quad \frac{1}{C_0} \leq \| F_\lambda^d \|_H \leq C_0 \quad \text{and} \quad \| \partial_d F_\lambda^d \|_H \leq \frac{C_0}{1 - d^2}.
\]

**Proof.** (i) Since we know by Proposition 1 and (31) that for any $(b, d) \in (-1, 1)^2$, the function

\[
G_{b,d}(y, s) = \kappa_0(1 - d^2)^{\frac{1}{p-1}} \left( (1 + be^s + dy)^{-\frac{2}{p-1}} \right)
\]

is a particular solution to the following vectorial form of Eq. (7):

\[
\frac{\partial}{\partial s} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \left( \mathcal{L} w_1 - \frac{2(p+1)}{(p-1)^2} w_1 + |w_1|^{p-1} w_1 - \frac{p+3}{p-1} w_2 - 2y \partial_y w_2 \right),
\]

it follows that $\partial_b G_{0,d}$ and $\partial_d G_{0,d}$ are particular solutions to the linearized equation around $G_{0,d} = \kappa(d, \cdot)$, which is precisely $\partial_1 w_1, w_2 = L_d(w_1, w_2)$ by definition of $L_d$ (100). Since we have from (112),

\[
\partial_b G_{0,d}(y, s) = \frac{-2\kappa_0 e^s}{p - 1} (1 - d^2)^{\frac{1}{p-1}} \left( \frac{(1 + dy)^{-\frac{2}{p-1} - 1}}{(1 + dy)^{\frac{2}{p-1} - 1}} \right),
\]

\[
\partial_d G_{0,d}(y, s) = \frac{\partial_d \kappa(d, y)}{0} = \frac{-2\kappa_0 (1 - d^2)^{\frac{1}{p-1} - 1}}{p - 1} \left( \frac{(y + d)(1 + dy)^{-\frac{2}{p-1} - 1}}{0} \right),
\]

this concludes the proof of (i).

(ii) We first give the following claim.
Claim 4.3. Consider for some $\alpha > -1$ and $\beta \in \mathbb{R}$ the following integral:

$$I(d) = \int_{-1}^{1} \frac{(1 - y^2)\alpha}{(1 + dy)^\beta} dy.$$

Then, there exists $K(\alpha, \beta) > 0$ such that the following holds for all $d \in (-1, 1)$:

(i) if $\alpha + 1 - \beta > 0$, then $\frac{1}{K} \leq I(d) \leq K$;
(ii) if $\alpha + 1 - \beta = 0$, then $\frac{1}{K} \leq I(d)/|\log(1 - d^2)| \leq K$;
(iii) if $\alpha + 1 - \beta < 0$, then $\frac{1}{K} \leq I(d)(1 - d^2)^{-(\alpha + 1) + \beta} \leq K$.

Proof. Since $I(d)$ is continuous, positive and even, it is enough to show the desired estimate as $d \to -1$. Note first that (i) follows from the Lebesgue theorem. For (ii) and (iii), we perform the following change of variables $y = 1 + \frac{d + 1}{d} z$ and write

$$I(d) = \frac{(1 + d)^{\alpha + 1 - \beta}}{(-d)^{\alpha + 1}} \int_{0}^{2d} \left(2 + \frac{d + 1}{d} z\right)^{\alpha} \frac{z^{\alpha}}{(1 + z)^{\beta}} dz.$$  \hfill (115)

In the case (iii), we just use the Lebesgue theorem to see that $I(d)(1 + d)^{-(\alpha + 1) + \beta} \to 2^\alpha \int_{0}^{\infty} \frac{z^\alpha}{(1 + z^{\beta})} dz$. In the case (ii), note that the integral in (115) behaves like $2^\alpha |\log(-2d)|$ to get the result and conclude the proof of Claim 4.3. \hfill $\square$

Using (46) together with the definition of $F^d_{\lambda,i}$ (110) and straightforward computations, we see that for $\lambda = 1$ or 0, $i = 1$ or 2, and $|d| < 1$,

$$|F^d_{\lambda,i}(y)| \leq C \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^{\frac{1}{p-1}}}, \quad |\partial_y F^d_{\lambda,i}(y)| \leq C \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^{\frac{1}{p-1} + 1}},$$

$$|\partial_d F^d_{\lambda,i}(y)| \leq C \frac{(1 - d^2)^{\frac{1}{p-1} - 1}}{(1 + dy)^{\frac{2}{p-1}}}, \quad |\partial^2_{d,y} F^d_{\lambda,i}(y)| \leq C \frac{(1 - d^2)^{\frac{1}{p-1} - 1}}{(1 + dy)^{\frac{2}{p-1} + 1}}.$$

Using this and Claim 4.3, we see that (111) holds. This concludes the proof of Lemma 4.2. \hfill $\square$

4.3. Non-negative directions of $L^*_d$ and corresponding projections for $L_d$

Let us now find the eigenfunctions of $L^*_d$ associated to the eigenvalues $\lambda = 1$ and $\lambda = 0$. 
Lemma 4.4 (Eigenfunctions of $L^*_d$ associated with the eigenvalues $\lambda = 1$ and $\lambda = 0$).

(i) (Existence) For all $|d| < 1$ and $\lambda \in \{0, 1\}$, there exists $W^d_\lambda \in \mathcal{H}$ continuous in terms of $d$ such that $L^*_d(W^d_\lambda) = \lambda W^d_\lambda$ where

\[
W^d_{1,2}(y) = c_1(d) \frac{1 - y^2}{(1 + dy)^{\frac{p-1}{p-1}}}, \quad W^d_{0,2}(y) = c_0(d) \frac{y + d}{(1 + dy)^{\frac{2}{p-1}+1}},
\]

(116)

$W^d_{\lambda,1}$ is uniquely determined by

\[
-Lr + r = \left(\lambda - \frac{p + 3}{p - 1}\right)r_2 - 2yr'_2 + \frac{8}{p - 1} \frac{r_2}{1 - y^2}
\]

with $r_2 = W^d_{\lambda,2}$ and the $C^1$ function $c_\lambda(d) > 0$ fixed by the relation

\[
\phi(W^d_\lambda, F^d_\lambda) = 1.
\]

(118)

(ii) (Orthogonality) For all $|d| < 1$ and $\lambda \in \{0, 1\}$, we have $\phi(W^d_\lambda, F^d_{1-\lambda}) = 0$.

(iii) (Normalization) There exists $C_0 > 0$ such that for $\lambda = 1$ or 0 and $|d| < 1$,

\[
\|W^d_\lambda\|_{\mathcal{H}} \leq C_0 \quad \text{and} \quad \|\partial_d W^d_\lambda\|_{\mathcal{H}} \leq \frac{C_0}{1 - d^2}.
\]

(119)

Proof. (ii) This is the standard orthogonality relation between eigenfunctions of $L_d$ and $L^*_d$ for different eigenvalues.

(i) We restrict ourselves to the proof of existence of $(W^d_{\lambda,1}, W^d_{\lambda,2})$ such that (116) and (117) hold with $c_\lambda(d) = 1$. Indeed:

- The fact that $W^d_{\lambda,1} \in \mathcal{H}$ will follow from (iii).
- The condition (118) follows directly from (116) and (117) as we show now.

Using (103) and (117), we write

\[
\phi(W^d_\lambda, F^d_\lambda) = \int_{-1}^{1} \left((LW^d_{\lambda,1} + W^d_{\lambda,1})F^d_{\lambda,1} + W^d_{\lambda,2}F^d_{\lambda,2}\right) \rho dy
\]

\[
= \int_{-1}^{1} \left(\left(\lambda - \frac{p + 3}{p - 1}\right)W^d_{\lambda,2} - 2yW^d_{\lambda,2} + \frac{8}{p - 1} \frac{W^d_{\lambda,2}}{1 - y^2}\right) F^d_{\lambda,1} \rho dy
\]

\[
+ \int_{-1}^{1} W^d_{\lambda,2} F^d_{\lambda,2} \rho dy.
\]

(120)

When $\lambda = 1$, we use (107), Lemma 4.2 (in particular the fact that $F^d_{1,1} = F^d_{1,2}$) and (116) to write
\[ \phi(W^d_1, F^d_1) = \int_{-1}^{1} W^d_{1,2} \left( \frac{3p+1}{p-1} F^d_{1,1} + 2yF^d_{1,1}' \right) \rho \, dy \]

\[ = c_1(d)(1 - d^2)^{\frac{p-1}{p-1}} \int_{-1}^{1} \frac{1 - y^2}{(1 + dy)^{\frac{2}{p-1} + 1}} \left( \frac{1 + dy + 2(p + 1)/(p - 1)}{1 + dy} \right)^{\frac{2}{p-1} + 2} \rho \, dy \]

which shows the integral of a positive function on \((-1, 1)\). Therefore, one can fix \(c_1(d)\) such that \(\phi(W^d_1, F^d_1) = 1\). Using Claim 4.3, we see that (121) holds. Therefore, one can fix \(c_1(d)\) such that \(\phi(W^d_1, F^d_1) = 1\). Using Claim 4.3, we see that for \(\lambda = 1\), the following holds:

\[ 0 < c_\lambda(d) \leq C(1 - d^2)^{\frac{1}{p-1}} \quad \text{and} \quad |c_\lambda'(d)| \leq C(1 - d^2)^{\frac{1}{p-1} - 1}. \quad (121) \]

When \(\lambda = 0\), we use (120), Lemma 4.2 and (116) (note in particular that \(W^d_{0,2}(y) = \frac{c_{0(d)}(d)}{1 - d^2} F^d_{0,1}(y)\)) to write

\[ \phi(W^d_0, F^d_0) = \frac{c_0(d)}{(1 - d^2)^{\frac{1}{p-1}}} \left[ \int_{-1}^{1} \left( \frac{-p + 3}{p - 1} + \frac{8}{(p - 1)(1 - y^2)} \right) \left( F^d_{0,1} \right)^2 \rho \, dy + \int_{-1}^{1} \left( F^d_{0,1} \right)^2 (y\rho)' \, dy \right] \]

\[ = \frac{c_0(d)}{(1 - d^2)^{\frac{1}{p-1}}} \int_{-1}^{1} \left( \frac{-p + 3}{p - 1} + \frac{8}{(p - 1)(1 - y^2)} + 1 - \frac{4y^2}{(p - 1)(1 - y^2)} \right) F^d_{0,1}^2 \rho \, dy \]

\[ = c_0(d)(1 - d^2)^{\frac{1}{p-1}} \int_{-1}^{1} \frac{(y + d)^2}{p - 1} \frac{\rho}{(1 + dy)^{\frac{2}{p-1} + 2}} \frac{1}{1 - y^2} \, dy \]

showing a positive integral. Therefore, one can fix \(c_0(d)\) such that \(\phi(W^d_0, F^d_0) = 1\). Using Claim 4.3, we see that (121) holds.

We now start the proof of the existence of \((W^d_{\lambda,1}, W^d_{\lambda,2})\) satisfying (116) and (117). The following claim allows us to conclude.

**Claim 4.5.**

(i) For any \(r_2 \in \mathcal{H}_0\), Eq. (117) has a unique solution \(r \in \mathcal{H}_0\) (10) such that

\[ \|r\|_{\mathcal{H}_0} \leq C\|r_2\|_{\mathcal{H}_0}. \quad (122) \]

(ii) For any \(|d| < 1\), \(\lambda \in \mathbb{R}\) and \(r \in \mathcal{H}_0\), we have the following equivalence: \(L^*_d(r) = \lambda r\) if and only if the function \(e^{-\lambda s} r_2(y)\) is a solution to the equation

\[ \partial^2_{ss} w = \mathcal{L} w + \psi(d, y) w - \frac{p + 3}{p - 1} \partial_s w - 2y\partial^2_{y,s} w + \frac{8}{p - 1} \frac{\partial_s w}{1 - y^2} \quad (123) \]

and \(r_1\) is a solution to (117).
Indeed, let us first use this claim to conclude the proof of (i) of Lemma 4.4. We first consider the case $d = 0$.

**Case $d = 0$.** One can check by hand that $e^{-s}(1 - y^2)$ and $y$ are solutions to (123) (one may use (27) when $\lambda = 0$). Therefore, from Claim 4.5, the function $(W_{\lambda,1}^0, W_{\lambda,2}^0)$ where $W_{1,2}(y) = 1 - y^2$, $W_{0,1}^0(y) = y$ and $W_{\lambda,2}^d$ is the unique solution of (117) with $r_2 = W_{\lambda,2}^d$ is an eigenfunction of $L_d^*$ corresponding to the eigenvalue $\lambda$.

**Case $d \neq 0$.** From the case $d = 0$, consider $(q_1, q_2) \in \mathcal{H}$ where

$$q_2(y) = 1 - y^2 \quad \text{(respectively } q_2(y) = y) \quad (124)$$

is an eigenfunction of $L_0^*$ corresponding to the eigenvalue $\lambda = 1$ (respectively $\lambda = 0$). If we introduce

$$w(y, s) = e^{-\lambda s} q_2(y), \quad (125)$$

then we see from (ii) of Claim 4.5 that $w$ is a solution to Eq. (123) with $d = 0$. If we introduce $W(Y, S) = T_d w$ defined by (33), then we see from Lemma 2.7 and the fact that

$$\frac{\partial_s w}{1 - y^2} = \frac{(1 + dY)^{\frac{2\mu}{\nu+1}}}{(1 - d^2)^{\frac{p}{\nu+1}}} \frac{\partial_s W}{1 - Y^2}$$

that $W(Y, S)$ satisfies Eq. (123) too. Since by (125), (33) and (124), we see that

$$W(Y, S) = \frac{(1 - d^2)^{\frac{1}{\nu+1}} w \left( \frac{Y + d}{1 + dY}, S - \log \frac{1 + dY}{\sqrt{1 - d^2}} \right)}{(1 + dY)^{\frac{2\mu}{\nu+1}}}$$

$$= \frac{(1 - d^2)^{\frac{1}{\nu+1}}}{1 + dY} e^{-\lambda S - \frac{1 + dY}{\sqrt{1 - d^2}}} q_2 \left( \frac{Y + d}{1 + dY} \right)$$

$$= e^{-\lambda S} \frac{(1 - d^2)^{\frac{1}{\nu+1}}}{1 + dY} q_2 \left( \frac{Y + d}{1 + dY} \right),$$

which is of the form $e^{-\lambda S} Q_2(Y)$ with

$$Q_2(Y) = q_2 \left( \frac{y + d}{1 + yd} \right) (1 + dy)^{\frac{2\mu}{\nu+1} - \lambda}$$

with

$$Q_2^d(y) = (1 - d^2) \frac{1 - y^2}{(1 + dy)^{\frac{2\mu}{\nu+1} + 1}} \quad \left( \text{respectively } Q_2^d(y) = \frac{y + d}{(1 + dy)^{\frac{2\mu}{\nu+1} + 1}} \right).$$
using (ii) of Claim 4.5, we see that \((Q_d^1, Q_d^2)\) where \(Q_d^1(y)\) is uniquely determined by Eq. (117) with \(r_2 = Q_d^2\) is an eigenvalue of \(L_d^*\) for the eigenvalue \(\lambda\). It remains to prove Claim 4.5 to conclude the proof of (i) of Lemma 4.4.

**Proof of Claim 4.5.** Note first that (ii) is classical and straightforward from the expression of \(L_d^*\) (104).

(i) If \(r_2 \in \mathcal{H}_0\) and

\[
f(y) = \left(\lambda - \frac{p + 3}{p - 1}\right) r_2 - 2yr'_2 + \frac{8r_2}{(p - 1)(1 - y^2)},
\]

then we write by Cauchy–Schwarz inequality and Hardy estimate (21) for all \(h \in \mathcal{H}_0\),

\[
\left| \int_{-1}^{1} f(y)h(y) \rho(y) dy \right| \leq C \|r_2\|_{L^p} \|h\|_{L^p} + C \left( \left\|r'_2 \sqrt{1 - y^2}\right\|_{L^p} + \left\|r_2 \sqrt{1 - y^2}\right\|_{L^p} \right) \left\|h \sqrt{1 - y^2}\right\|_{L^p}.
\]

Therefore, the linear form \(h \rightarrow \int_{-1}^{1} f(y)h(y) \rho(y) dy\) is in the dual of \(\mathcal{H}_0\) and \(\|f\|_{\mathcal{H}'_0} \leq C \|r_2\|_{\mathcal{H}_0}\). Since \(\mathcal{H}_0\) equipped with the inner product defined in (109) is a Hilbert space, there is a unique \(r \in \mathcal{H}_0\) such that

\[
\forall h \in \mathcal{H}_0, \quad \langle r, h \rangle_{\mathcal{H}_0} = \int_{-1}^{1} f(y)h(y) \rho(y) dy \quad \text{and} \quad \|r\|_{\mathcal{H}_0} \leq \|f\|_{\mathcal{H}'_0} \leq C \|r_2\|_{\mathcal{H}_0}\). (127)

Using (109), we see that \(r) is the unique solution of Eq. (117), and (122) follows from (127). This concludes the proof of Claim 4.5. \(\square\)

(iii) **(Normalization)** Since \(W_{\lambda,1}^d\) and \(\partial_d W_{\lambda,1}^d\) are solutions to Eq. (117) respectively with \(r_2 = W_{\lambda,2}^d\) and \(r_2 = \partial_d W_{\lambda,2}^d\), we see from (i) in Claim 4.5 that for \(\lambda = 1\) or 0 and \(|d| < 1\),

\[
\|W_{\lambda}^d\|_{\mathcal{H}} \leq C_0 \|W_{\lambda,2}^d\|_{\mathcal{H}_0} \quad \text{and} \quad \|\partial_d W_{\lambda,2}^d\|_{\mathcal{H}} \leq C_0 \|\partial_d W_{\lambda,2}^d\|_{\mathcal{H}_0}. (128)
\]

Using (46) together with the definition of \(W_{l,2}^d\), (121) and straightforward computations, we see that for \(\lambda = 1\) or 0 and \(|d| < 1\),

\[
\|W_{\lambda,2}^d(y)\| \leq C \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^{\frac{2}{p-1}}}, \quad \|\partial_y W_{\lambda,2}^d(y)\| \leq C \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^{\frac{2}{p-1} + 1}},
\]

\[
\|\partial_d W_{\lambda,2}^d(y)\| \leq C \frac{(1 - d^2)^{\frac{1}{p-1} - 1}}{(1 + dy)^{\frac{2}{p-1}}}, \quad \|\partial_{d,y} W_{\lambda,2}^d(y)\| \leq C \frac{(1 - d^2)^{\frac{1}{p-1} - 1}}{(1 + dy)^{\frac{2}{p-1} + 1}}.
\]
Since we have by this, by Claim 4.3 and by the definition of the norm in $\mathcal{H}_0$, $\|W^d_{\lambda,2}\|_{\mathcal{H}_0} + (1 - d^2)\|\partial_d W^d_{\lambda,2}\|_{\mathcal{H}_0} \leq C_0$, we see that (119) follows by (128). This concludes the proof of Lemma 4.4. □

4.4. Expansion of $q$ with respect to the eigenspaces of $L_d$

In the following, we expand any $q \in \mathcal{H}$ with respect to the eigenspaces of $L_d$ partially computed in Lemma 4.2. We claim the following.

**Definition 4.6** (Expansion of $q$ with respect to the eigenspaces of $L_d$). Consider $q \in \mathcal{H}$ and introduce for $\lambda = 1$ and $\lambda = 0$

$$\pi^d_{\lambda}(q) = \phi(W^d_{\lambda}, q), \tag{129}$$

where $W^d_{\lambda}$ is the eigenfunction of $L^*_d$ computed in Lemma 4.4, and $\pi^d_{-}(q) = q_{-}$ defined by

$$q = \pi^d_1(q)F^d_1(y) + \pi^d_0(q)F^d_0(y) + \pi^d_-(q). \tag{130}$$

Applying the operator $\pi^d_{\lambda}$ to (130), we write

$$\pi^d_{\lambda}(q) = \pi^d_{\chi}(q)\pi^d_{\lambda}(F^d_{\lambda}) + \pi^d_{1-\lambda}(q)\pi^d_{\lambda}(F^d_{1-\lambda}) + \pi^d_{\lambda}(\pi^d_{-}(q)).$$

Since

$$\pi^d_{\lambda}(F^d_{\mu}) = \delta_{\lambda,\mu} \tag{131}$$

by (118) and (ii) of Lemma 4.4, this yields

$$\phi(F^d_{\lambda}, q_{-}) = \pi^d_{\lambda}(\pi^d_{-}(q)) = 0. \tag{132}$$

Therefore, we have

$$\pi^d_{-}(q) \in \mathcal{H}^d_{-} = \{r \in \mathcal{H} | \pi^d_1(r) = \pi^d_0(r) = 0\}. \tag{133}$$

**Remark.** Note that if $q \in \mathcal{H}^d_{-}$, then $\pi^d_{-}(q) = q$ (just use (130) and (133)) and $L_d q \in H^d$. Indeed, using the definition of $\pi^d_{\lambda}$ (129), (106) and Lemma 4.4, we write $\pi^d_{\lambda}(L_d q) = \phi(W^d_{\lambda}, L_d q) = \phi(L^*_d W^d_{\lambda}, q) = \phi(\lambda W^d_{\lambda}, q) = \lambda \pi^d_{\lambda}(q) = 0$. Moreover $\pi^d_{-}(F^d_{\lambda}) = 0$ for $\lambda = 0$ or 1 (just use (130) with $q = F^d_{\lambda}$ and (131)).

**Remark.** Note that $\pi^d_{\lambda}(q)$ is the projection of $q$ on the eigenfunction of $L_d$ associated to $\lambda$, and that $\pi^d_{-}(q)$ is the negative part of $q$. 
4.5. Equivalent norms on $\mathcal{H}$ and $\mathcal{H}^d_-$ adapted to the dispersive structure

For the proof of the main theorem, we will need to prove in some sense dispersive estimates on $q_- = \pi^d_d(q)$ when $q$ is a solution to (99). In order to achieve this, we need to manipulate a function of $q_-$ (equivalent to the norm $\|q_-\|_{\mathcal{H}} = \phi(q_-, q_-)^{1/2}$ in $\mathcal{H}_-$) which will capture the dispersive character of Eq. (99). Such a quantity will be

\[ \varphi_d(q, r) = \int_{-1}^{1} (-\psi(d, y)q_1r_1 + q'_1r'_1(1 - y^2) + q_2r_2) \rho \, dy \]  

\[ = \int_{-1}^{1} (-q_1(\mathcal{L}r_1 + \psi(d, y)r_1) + q_2r_2) \rho \, dy, \]  

(134)

where $\psi(d, y)$ is defined in (100). This bilinear form is in fact the second variation of $E(w(s))$ defined in (141) around $\kappa(d, y)$ (13), the stationary solution of (7), and can be seen as the energy norm in $\mathcal{H}^d_-$ (space where it will be definite positive). More precisely, we have the following.

**Proposition 4.7 (Equivalence in $\mathcal{H}^d_-$ of the $\mathcal{H}$ norm and the energy norm).** There exists $C_0 > 0$ such that for all $|d| < 1$, the following holds:

(i) Equivalence of norms in $\mathcal{H}^d_-$. For all $q_- \in \mathcal{H}^d_-$,

\[ \frac{1}{C_0} \|q_-\|_{\mathcal{H}}^2 \leq \varphi_d(q_-, q_-) \leq C_0 \|q_-\|_{\mathcal{H}}^2. \]

(ii) Equivalence of norms in $\mathcal{H}$. For all $q \in \mathcal{H}$,

\[ \frac{1}{C_0} \|q\|_{\mathcal{H}} \leq (|\pi^d_1(q)| + |\pi^d_0(q)| + \sqrt{\varphi_d(q_-, q_-)}) \leq C_0 \|q\|_{\mathcal{H}}, \]

where $\varphi_d$ is given in (134) and $q$ is expanded as in (130).

**Remark.** Note that $\varphi_d$ is not positive in $\mathcal{H}$ (for example, $\varphi_d((1, 0), (1, 0)) = -\int \psi \rho \, dy < 0$). In particular, its quadratic form cannot be considered as a norm in $\mathcal{H}$. However, we will show that it is definite positive on the space $\mathcal{H}^d_-$, uniformly for $|d| < 1$, which gives the control of the norm by $\varphi_d$ (independent of $d$). A remarkable fact is that the constant $C_0$ is independent of $d$. In the following, we reduce the proof of Proposition 4.7 to the proof of the fact that the following approximation of $\varphi_d$ defined for $\epsilon > 0$ is non-negative:

\[ \varphi_{d, \epsilon}(q, r) = (1 - \epsilon) \int_{-1}^{1} q'_1r'_1(1 - y^2) \rho \, dy \]

\[ + \int_{-1}^{1} \left( \left( -(1 - \epsilon)\psi(d, y) - \epsilon \frac{3p + 1}{p - 1} \frac{(1 - d^2)}{(1 + dy)^2} \right) q_1r_1 + (1 - \epsilon)q_2r_2 \right) \rho \, dy \]  

(136)
\[
\begin{align*}
&= \int_{-1}^{1} q_1 \left( -(1 - \epsilon) L r_1 + \left( -(1 - \epsilon) \psi(d, y) - \epsilon \frac{3p + 1}{p - 1} \frac{(1 - d^2)}{(1 + dy)^2} r_1 \right) \rho \, dy \\
&\quad + (1 - \epsilon) \int_{-1}^{1} q_2 r_2 \rho \, dy.
\end{align*}
\]

We claim that the following lemma directly implies Proposition 4.7.

**Lemma 4.8** *(Reduction of the proof of Proposition 4.7).* There exists \( \epsilon_0 \in (0, 1) \) such that for all \(|d| < 1 \) and \( q_- \in \mathcal{H}^d_- \), \( \varphi_{d, \epsilon_0}(q_-, q_-) \geq 0 \) where \( \varphi_{d, \epsilon_0} \) is defined in (136).

**Remark.** One could choose other approximations of \( \varphi_d \), but our choice (136) is particularly well adapted for the proof, as it gives a simple form after the Lorentz transform in similarity variables given in Lemma 2.6. See the proof of Lemma 4.10 below.

Indeed, let us first assume Lemma 4.8 and prove Proposition 4.7.

**Lemma 4.8 implies Proposition 4.7.**

*Proof of (i).* For the upper bound, just note that since we easily have

\[
\frac{(1 - d^2)(1 - y^2)}{(1 + dy)^2} \leq 1, \quad \text{hence} \quad |\psi(d, y)| \leq \frac{C}{1 - y^2}
\]

we see from the definitions of \( \varphi_d \) (134) and the Hardy–Sobolev estimate (21) that for any \(|d| < 1 \) and \( q \) and \( r \) in \( \mathcal{H} \),

\[
|\varphi_d(q, r)| \leq \|q\|_{\mathcal{H}}\|r\|_{\mathcal{H}} + C \left\| \frac{1}{\sqrt{1 - y^2}} \right\|_{L^2_p} \left\| \frac{r_1}{\sqrt{1 - y^2}} \right\|_{L^2_p} \leq C_0\|q\|_{\mathcal{H}}\|r\|_{\mathcal{H}}. \quad (138)
\]

For the lower bound, fix \( \epsilon = \epsilon_0 \) defined in Lemma 4.8, take \(|d| < 1 \), \( q_- \in \mathcal{H}^d_- \) and write

\[
0 \leq \varphi_{d, \epsilon_0}(q_-, q_-) = \varphi_d(q_-, q_-) - \epsilon_0 \int_{-1}^{1} \left( q'_{-1}^2 (1 - y^2) + q_{-2}^2 + g(d, y) q_{-1}^2 \right) \rho \, dy, \quad (139)
\]

where

\[
g(d, y) = \frac{3p + 1}{p - 1} \frac{(1 - d^2)}{(1 + dy)^2} - \psi(d, y) = \frac{p + 1}{p - 1} \frac{(1 - d^2)}{(1 + dy)^2} + \frac{2(p + 1)}{(p - 1)^2} \geq \frac{2(p + 1)}{(p - 1)^2}
\]

(see (41)). Therefore,

\[
\varphi_d(q_-, q_-) \geq \alpha_0 \epsilon_0 \int_{-1}^{1} \left( q'_{-1}^2 (1 - y^2) + q_{-1}^2 + q_{-2}^2 \right) \rho \, dy = \alpha_0 \epsilon_0 \|q_-\|_{\mathcal{H}}^2
\]
for some positive $\alpha_0$ which is the conclusion of (i).

Proof of (ii). Using the definition of $\phi$ (102) and (130), we write

$$\|q\|_{\mathcal{H}}^2 = \phi(q, q) = (\pi_{\lambda}^d(q))^2 \|F_{\lambda}^d\|_{\mathcal{H}}^2 + (\pi_{\lambda}^d(q))^2 \|F_{\lambda}^d\|_{\mathcal{H}}^2 + \|q_\perp\|_{\mathcal{H}}^2.$$  

Using (111), we get the following equivalence of norms:

$$\frac{1}{C} \|q\|_{\mathcal{H}} \leq \sum_{\lambda=0}^{1} |\pi_{\lambda}^d(q)| + \|q_\perp\|_{\mathcal{H}} \leq C \|q\|_{\mathcal{H}}. \quad (140)$$

Since $q_\perp \in \mathcal{H}_\perp^d$ by (133), we can use (i) to conclude. This concludes the proof of proposition assuming Lemma 4.8. □

Let us now prove Lemma 4.8.

Proof of Lemma 4.8. We proceed in 3 parts:

- In Part 1, we find a subspace of $\mathcal{H}$ of codimension 2 where $\varphi_{d, \epsilon}$ is non-negative.
- In Part 2, we find a plane in $\mathcal{H}$, where $\varphi_{d, \epsilon}$ is negative and which is orthogonal to $\mathcal{H}_\perp^d$ with respect to $\varphi_{d, \epsilon}$.
- In Part 3, we proceed by contradiction and prove that $\varphi_{d, \epsilon}$ is non-negative on $\mathcal{H}_\perp^d$.

Part 1. $\varphi_{d, \epsilon}$ is non-negative on a subspace of codimension 2. We claim the following.

Lemma 4.9. There exists $\epsilon_1 > 0$ such that for all $|d| < 1$ and $\epsilon \in (0, \epsilon_1]$, $\varphi_{d, \epsilon}$ is non-negative on the subspace

$$E_2 = \left\{ q \in \mathcal{H} \mid \int_{-1}^{1} T_{-d}(q_1) \rho(y) dy = \int_{-1}^{1} T_{-d}(q_1) y \rho(y) dy = 0 \right\}, \quad (141)$$

where $T_{-d}$ is defined in (33).

Proof. Define from (26) $\epsilon_1 = \frac{\gamma_1 - \gamma_2}{1 - \gamma_2} > 0$ and fix $\epsilon \in (0, \epsilon_1]$. We consider $(u_1, u_2) \in E_2$, and write by (137),

$$\varphi_{d, \epsilon}(u, u) = \int_{-1}^{1} u_1 \left( -(1 - \epsilon) L u_1 + \left[ -(1 - \epsilon) \psi(d, y) - \epsilon \frac{3p + 1}{p - 1} \frac{(1 - d^2)}{(1 + dy)^2} \right] u_1 \right) \rho(y) dy$$

$$+ (1 - \epsilon) \int u_2^2 \rho(y) dy. \quad (142)$$

If $U_1 = T_{-d} u_1$, then $u_1 = T_d U_1$ and we have by (33) and (42),
\[ u_1(y) = \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^{\frac{2}{p-1}}} U_1(z) \quad \text{with} \quad z = \frac{y + d}{1 + dy}, \]

\[ \mathcal{L}u_1(y) + \psi(d, y)u_1(y) = \frac{(1 - d^2)^{\frac{1}{p-1}+1}}{(1 + dy)^{\frac{2}{p-1}+2}} \left( \mathcal{L}U_1(z) + \frac{2(p + 1)}{p - 1} U_1(z) \right), \]

\[ \rho(y) dy = \frac{(1 + dy)^{\frac{2(p + 1)}{p - 1}}}{(1 - d^2)^{\frac{1}{p-1}}} \rho(z) dz, \]

\[ 0 = \int U_1(z) \rho(z) dz = \int U_1(z) z \rho(z) dz. \quad (143) \]

Therefore, we see by (142) and Lemma 2.4 (use (143)) that

\[ \phi_{d,\epsilon}(u, u) = \int_{-1}^{1} U_1 \left( - (1 - \epsilon) \mathcal{L}U_1 - \left( \frac{2(p + 1)}{p - 1} + \epsilon \right) U_1 \right) \rho(z) dz \]

\[ + (1 - \epsilon) \int_{-1}^{1} u_2^2 \rho(y) dy \]

\[ \geq - (1 - \epsilon) \gamma_2 + (\gamma_1 - \epsilon) \int_{-1}^{1} U_1^2 \rho dy + (1 - \epsilon) \int_{-1}^{1} u_2^2 \rho(y) dy \geq 0 \]

since \(\epsilon \leq \epsilon_1\) hence \(- (1 - \epsilon) \gamma_2 + (\gamma_1 - \epsilon) \geq 0\). This concludes the proof of Lemma 4.9. \(\square\)

**Part 2.** \(\varphi_{d,\epsilon}\) is negative on a plane orthogonal to \(\mathcal{H}_d^d\). We need to find \(V_0^{d,\epsilon}\) and \(V_1^{d,\epsilon}\) linearly independent in \(\mathcal{H}\) such that \(\varphi_{d,\epsilon}(V_0^{d,\epsilon}, r) = 0\) for any \(r \in \mathcal{H}_d^d\). Since we know by the definition of \(\mathcal{H}_d^d\) (133), that

\[ \forall r \in \mathcal{H}_d^d, \quad \phi(W_1^d, r) = \pi_1^d(r) = 0 \quad \text{and} \quad \phi(W_0^d, r) = \pi_0^d(r) = 0, \]

a convenient way to conclude is to find \(V_1^{d,\epsilon}\) and \(V_0^{d,\epsilon}\) such that

\[ \forall q \in \mathcal{H}, \quad \phi(W_1^d, q) = \varphi_{d,\epsilon}(V_1^{d,\epsilon}, q) \quad \text{and} \quad \phi(W_0^d, q) = \varphi_{d,\epsilon}(V_0^{d,\epsilon}, q). \quad (144) \]

Then, we will show that \(\varphi_{d,\epsilon}\) is negative on the plane spanned by \(V_1^{d,\epsilon}\) and \(V_0^{d,\epsilon}\). Consider \(\epsilon > 0\) going to zero and take \(|d| < 1\). We claim the following.

**Lemma 4.10.** There exists \(\epsilon_2 > 0\) such that for all \(\epsilon \in (0, \epsilon_2]\) and \(|d| < 1\):

(i) There exist continuous functions \(V_\lambda^{d,\epsilon}\) for \(\lambda \in \{0, 1\}\) such that (144) holds.
(ii) Moreover, it holds that
\[
\sup_{|d|<1} \left\| V_{1}^{d, \epsilon}(y) - \left( -\frac{W_{1,2}^{d}(y)}{W_{1,2}^{d}(y)} \right) - \alpha_1(d) F_0^d(y) \right\|_{\mathcal{H}_0} + \left\| \epsilon V_0^{d, \epsilon}(y) + \alpha_2 F_0^d(y) \right\|_{\mathcal{H}_0} \to 0 \tag{145}
\]
as \epsilon \to 0^+ where \( \alpha_1(d) \) is continuous, \( \alpha_2 > 0 \), \( W_{1,2}^{d} \) and \( F_0^d \) are defined in (116) and (110).

(iii) The bilinear form \( \varphi_{d, \epsilon} \) is negative on the plane of \( \mathcal{H} \) spanned by \( V_0^{d, \epsilon} \) and \( V_1^{d, \epsilon} \).

**Remark.** Note that in this lemma, we find explicit solutions for \( V_{\lambda}^{d, \epsilon} \) which was not the case for KdV and NLS (see [13] and [14]).

**Proof of Lemma 4.10.** We proceed in 3 steps:

- In Step 1, we find a PDE satisfies by \( V_{\lambda}^{d, \epsilon} \) and transform it with the Lorentz transform in similarity variables defined in (33).
- In Step 2, we solve the transformed PDE and find the asymptotic behavior of \( V_{\lambda}^{d, \epsilon} \) as \( \epsilon \to 0 \), uniformly in \( |d| < 1 \), which gives (i) and (ii).
- In Step 3, we use that asymptotic behavior to show that \( \varphi_{d, \epsilon} \) is negative on the plane spanned by \( V_1^{d, \epsilon} \) and \( V_0^{d, \epsilon} \), which gives (iii).

**Step 1. Reduction to the solution of some PDE.** (i) From the definitions of \( \varphi_{d, \epsilon} \) (137) and \( \phi \) (103), we see that in order to satisfy (144), it is enough to take
\[
V_{\lambda}^{d, \epsilon} = W_{\lambda,2}^d/(1 - \epsilon) \tag{146}
\]
and to prove the existence of \( V_{\lambda,1}^{d, \epsilon} \) solution to
\[
-(1 - \epsilon) \mathcal{L} V_{\lambda,1}^{d, \epsilon} + \left( -(1 - \epsilon) \psi(d, y) - \epsilon \frac{3p + 1}{p - 1} \frac{(1 - d^2)}{(1 + dy)^2} \right) V_{\lambda,1}^{d, \epsilon} = -\mathcal{L} W_{\lambda,1}^d + W_{\lambda,1}^d. \tag{147}
\]

In the following, we use the Lorentz transform (33) and transform this equation to make it ready to solve using the spectral properties of \( \mathcal{L} \) stated in Proposition 2.3. More precisely, we have the following.

**Claim 4.11 (Reduction to an explicitly solvable PDE).** Consider \( V_{\lambda,1}^{d, \epsilon} \) and introduce \( \tilde{v}_{\lambda,1}^{d, \epsilon} \) defined by
\[
\tilde{v}_{\lambda,1}^{d, \epsilon} = \mathcal{T}_{-d} V_{\lambda,1}^{d, \epsilon}, \tag{148}
\]
where \( \mathcal{T}_{-d} \) is defined in (33). Then,

(i) \( V_{\lambda,1}^{d, \epsilon} \) is a solution to (147) if and only if \( \tilde{v}_{\lambda,1}^{d, \epsilon} \) is a solution to the equation
\[
(1 - \epsilon) \mathcal{L} \tilde{v}_{\lambda,1}^{d, \epsilon}(z) + (-\gamma_1 + \epsilon) \tilde{v}_{\lambda,1}^{d, \epsilon}(z) = f_{d}^{1} = \frac{1 - d^2}{(1 - dz)^2} \mathcal{T}_{-d} (\mathcal{L} W_{\lambda,1}^d - W_{\lambda,1}^d). \tag{149}
\]
and \( \gamma_1 = -\frac{2(p+1)}{p-1} \) is defined in (26).

(ii) The linear form \( h \to \int_{-1}^1 f^d_\lambda h \rho \) is continuous on \( \mathcal{H}_0 \) and for some \( C_0 > 0 \), we have

\[
\forall d \in (-1, 1), \quad \| f^d_\lambda \|_{\mathcal{H}_0} \leq C_0 \| W^d_\lambda \|_{\mathcal{H}} \leq C_0^2.
\]

**Proof.** (i) Using (33) and Lemma 2.6, we see that

\[
V^{d,e}_{\lambda,1}(y) = \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^\frac{2}{p-1}} v^{d,e}_{\lambda,1}(z) \quad \text{with} \quad z = \frac{y + d}{1 + dy},
\]

\[
\mathcal{L}V^{d,e}_{\lambda,1}(y) + \psi(d,y)V^{d,e}_{\lambda,1}(y) = \frac{(1 - d^2)^{\frac{1}{p-1} + 1}}{(1 + dy)^\frac{2}{p-1} + 2} \left( \mathcal{L}v^{d,e}_{\lambda,1}(z) + \frac{2(p+1)}{p - 1} v^{d,e}_{\lambda,1}(z) \right).
\]

(150)

Since \( \frac{(1-dz)^2}{1-d^2} = \frac{1-d^2}{(1+dy)^2} \) and \( \gamma_1 = -\frac{2(p+1)}{p-1} \) (see (26)), we see that Eqs. (147) and (149) are equivalent.

(ii) Note from (33) that for all \( V_1 \) and \( V_2 \) in \( L^2_\rho \),

\[
\int_{-1}^1 V_1(Y)V_2(Y)\rho(Y)\,dy = \int_{-1}^1 \frac{1 - d^2}{(1 - dy)^2} v_1(y)v_2(y)\rho(y)\,dy,
\]

(151)

where \( v_i = T_d V_i \). Therefore, using (149) and (151), we have for any \( h \in \mathcal{H}_0 \),

\[
\int_{-1}^1 f^d_\lambda(z)h(z)\rho(z)\,dy = \int_{-1}^1 (\mathcal{L}W^d_{\lambda,1} - W^d_{\lambda,1})H\rho = \int_{-1}^1 (\partial_y W^d_{\lambda,1}\partial_y H(1 - y^2) + W^d_{\lambda,1}H)\rho,
\]

where \( H = T_d h \). Therefore, using the continuity of \( T_d \) in \( \mathcal{H}_0 \) (see Lemma 2.8) and the bound on \( \| W^d_\lambda \|_{\mathcal{H}} \) (119), we see that

\[
\left| \int_{-1}^1 f^d_\lambda(z)h(z)\rho(z)\,dy \right| \leq \| W^d_{\lambda,1} \|_{\mathcal{H}_0} \| H \|_{\mathcal{H}_0} \leq C_0 \| W^d_\lambda \|_{\mathcal{H}} \| h \|_{\mathcal{H}_0} \leq C_0^2 \| h \|_{\mathcal{H}_0},
\]

which closes the proof of Claim 4.11. \( \Box \)

**Step 2.** Solution of Eq. (149) and asymptotic behavior as \( \epsilon \to 0 \). We prove (i) and (ii) of Lemma 4.10 in this step.

**Proof of Lemma 4.10(i).** Note first that since

\[
T_d(z) = F^d_{0,1}
\]

(152)
by definition of $T_d$ (33) and $F_0^d$ (110), we have from the definition of $f^d_\lambda$ (149), (151), the expression of $\phi$ (103) and Lemma 4.4 the following: for all $|d| < 1$,

$$
\int_{-1}^{1} f^d_\lambda(z)z\rho(z)\,dz = \int_{-1}^{1} \left( LW^d_{\lambda,1}(y) - W^d_{\lambda,1}(y) \right) F^d_{0,1}(y)\rho(y)\,dy = -\delta_{\lambda,0}.
$$

(153)

We have the following claim which is a consequence of Proposition 2.3.

**Claim 4.12 (Solution of Eq. (149)).** Consider

$$
f = \sum_{n=0}^{\infty} f_n h_n(y) \in H'_0
$$

where $h_n$ are the eigenfunctions of $L$ defined in Proposition 2.3. Then, for any $\epsilon \in (0, \frac{1}{2})$, the following equation:

$$
(1 - \epsilon)Lv + (-\gamma_1 + \epsilon)v = f
$$

(154)

has a unique solution in $H_0$ given by

$$
v = \sum_{n=0}^{\infty} \frac{f_n}{\gamma_n - \gamma_1 + \epsilon(1 - \gamma_n)} h_n,
$$

(155)

where $\gamma_n \leq 0$ are the eigenvalues of $L$ introduced in Proposition 2.3.

From this claim and (ii) in Claim 4.11, we see that for all $\epsilon \in (0, \frac{1}{2})$, $|d| < 1$ and $\lambda = 1$ or $\lambda = 0$, Eq. (149) has a solution $v_{d,\epsilon}^{\lambda,1}$. Using (i) in Claim 4.11, we see that Eq. (147) has a solution $V_{\lambda,1}^{d,\epsilon}$ given by (148), which closes the proof of Lemma 4.10(i).

**Proof of Lemma 4.10(ii).** When $\lambda = 1$, we see from (153), (29) and (27) that

$$
(f_1^d)_1 = \int_{-1}^{1} f_1^d(z)z\rho(z)\,dz = 0.
$$

Therefore, we see from Claim 4.12 and the definition of $f_1^d$ (149) that for $\epsilon$ small enough,

$$
\sup_{|d| < 1} \left\| v_{1,1}^{d,\epsilon} - v^* \right\|_{H_0} \leq C \epsilon \left\| f_1^{d,\epsilon} \right\|_{H'_0} \leq C_0 \epsilon \text{ where } v^*(z) = \sum_{n \neq 1} \frac{(f_1^d)_n}{\gamma_n - \gamma_1} h_n(z)
$$

is the unique solution of

$$
L v(z) - \gamma_1 v(z) = f_1^d(z) \text{ with } \int_{-1}^{1} v(z)z\rho(z)\,dz = 0.
$$

(156)
Therefore, we see from (148) and Lemma 2.8 that for \( \epsilon \) small enough,

\[
\sup_{|d|<1} \left\| V_{1,1}^{d,\epsilon} - V^* \right\|_{\mathcal{H}_0} \leq C_0 \epsilon, \tag{157}
\]

where \( V^* = T_d v^* \) is the unique solution of

\[
\mathcal{L} V(y) + \psi(d, y)V(y) = \mathcal{L} W_{1,1}^d - W_{1,1}^d \quad \text{with} \quad \int_{-1}^{1} V(y) F_{0,1}^{d}(y) \frac{\rho(y)}{(1 + dy^2)} dy = 0
\]

(note that this equation is the version of (147) with \( \epsilon = 0 \) and use (151) together with (152) to get the orthogonality condition). Since

\[
-\mathcal{L} W_{1,1}^d + W_{1,1}^d = \mathcal{L} W_{1,2}^d + \psi(d, y) W_{1,2}^d \quad \text{and} \quad \mathcal{L} F_{0,1}^{d} + \psi(d, y) F_{0,1}^{d} = 0
\]

(see the fact that \( L_y^d(W_{1}^{d}) = W_{1}^{d} \) and \( L_d(F_{0}^{d}) = 0 \) from Lemmas 4.4 and 4.2), we see from the uniqueness that \( V^*(y) = -W_{1,2}(y) + \alpha_1(d) F_{0,1}^{d}(y) \) where

\[
\alpha_1(d) = \int_{-1}^{1} W_{1,2}^d(y) F_{0,1}^{d}(y) \frac{\rho(y)}{(1 + dy^2)} dy / \int_{-1}^{1} F_{0,1}^{d}(y)^2 \frac{\rho(y)}{(1 + dy^2)} dy
\]

is continuous. Thus, the first identity in (145) follows from (157), (146) and (116).

When \( \lambda = 0 \), we see from (153), (29) and (27) that

\[
\left( \tilde{f}_0^{d} \right)_1 = \int_{-1}^{1} f_0^{d}(z) z \rho(z) dz = -1.
\]

Therefore, since \( h_1(y) = c_1 y \) by (27), we see from Claim 4.12 and Claim 4.11(ii) that for \( \epsilon \) small enough,

\[
\left\| \tilde{v}_{0,1}^{d,\epsilon}(z) + \frac{\alpha_2}{\epsilon} z \right\|_{\mathcal{H}_0} \leq C \left\| f_0^{d} \right\|_{\mathcal{H}_0} \leq C_0 \quad \text{where} \quad \alpha_2 = \frac{1}{(1 - \gamma_1) \int_{-1}^{1} y^2 \rho(y) dy} > 0 \tag{158}
\]

(note from (26) that \( \gamma_1 = -\frac{2(p+1)}{p-1} < 0 \)). Since the estimate for \( V_{1,2}^{d,\epsilon} \) follows from (146) and (116), we see that (145) follows from (158), (148) and (152). This closes the proof of (i) and (ii) in Lemma 4.10.

**Step 3.** Sign of \( \varphi_{d,\epsilon} \) on the plane spanned by \( V_{1}^{d,\epsilon} \) and \( V_{0}^{d,\epsilon} \).
Proof of Lemma 4.10(iii). We finish the proof of Lemma 4.10 here, by proving that \( \varphi_{d,\epsilon} \) is negative on the plane of \( \mathcal{H} \) spanned by \( V^d_{0,\epsilon} \) and \( V^d_{1,\epsilon} \). It is enough to find \( \epsilon_4 \) such that for all \( 0 < \epsilon \leq \epsilon_4 \) and \( |d| < 1 \),

\[
\varphi_{d,\epsilon}(V^d_{0,\epsilon}, V^d_{0,\epsilon}) < 0 \quad \text{and} \quad \left| \varphi_{d,\epsilon}(V^d_{1,\epsilon}, V^d_{1,\epsilon}) - \varphi_{d,\epsilon}(V^d_{1,\epsilon}, V^d_{0,\epsilon}) \right| > 0.
\] (159)

In the following, we will estimate \( \varphi_{d,\epsilon}(V^d_{\lambda,\mu}, V^d_{\mu,\mu}) \) as \( \epsilon \to 0^+ \), uniformly for \( |d| < 1 \), using the asymptotic behavior of \( V^d_{\lambda,\mu} \) given in (145).

First, using (144) and the expression of \( \phi \) (103), we write

\[
\varphi_{d,\epsilon}(V^d_{\lambda,\mu}, V^d_{\mu,\mu}) = \phi(V^d_{\lambda,\mu}, W^d_{\mu,\mu})
\]

for \( \lambda, \mu \in \{0, 1\} \). Since \( \phi(F^d_{\lambda,\mu}, W^d_{\mu,\mu}) = \delta_{\lambda,\mu} \) by Lemma 4.4, taking \( \lambda = 0 \) and \( \mu \in \{0, 1\} \), we have from (145) and the continuity of \( \phi \) in \( H \) that

\[
\sup_{|d| \leq d_0} \left| \epsilon \varphi_{d,\epsilon}(V^d_{0,\epsilon}, V^d_{\mu,\mu}) + \alpha_2 \delta_{0,\mu} \right| \to 0 \quad \text{as } \epsilon \to 0.
\] (160)

Now, taking \( \lambda = \mu = 1 \), we see from (145) that

\[
\sup_{|d| \leq d_0} \left| \varphi_{d,\epsilon}(V^d_{1,\epsilon}, V^d_{1,\epsilon}) - \phi(W^d_{1,1}, \left(\frac{-W^d_{1,2}}{W^d_{1,2}}\right)) \right| \to 0 \quad \text{as } \epsilon \to 0.
\] (161)

Using (103) again together with (117), we write

\[
\phi\left(W^d_{1,1}, \left(\frac{-W^d_{1,2}}{W^d_{1,2}}\right)\right)
\]

\[
= \int_{-1}^{1} W^d_{1,2}(y)\left(L W^d_{1,1}(y) - W^d_{1,1}(y) + W^d_{1,2}(y)\right)\rho(y) \, dy
\]

\[
= \int_{-1}^{1} W^d_{1,2}(y)\left(\frac{p + 3}{p - 1} W^d_{1,1}(y) + 2y W^d_{1,2}(y) - \frac{8}{p - 1} \frac{W^d_{1,2}(y)}{1 - y^2}\right)\rho(y) \, dy
\]

\[
= \int_{-1}^{1} \frac{W^d_{1,2}(y)^2}{p - 1} \left(p + 3 - \frac{8}{1 - y^2}\right)\rho(y) \, dy - \int_{-1}^{1} W^d_{1,2}(y)^2(y\rho(y))' \, dy
\]

\[
= -\frac{4}{p - 1} \int_{-1}^{1} W^d_{1,2}(y)^2 \frac{\rho(y)}{1 - y^2} \, dy.
\] (162)

Using (160), (161) and (162), we see that

\[
\varphi_{d,\epsilon}(V^d_{0,\epsilon}, V^d_{0,\epsilon}) \sim -\frac{\alpha_2}{\epsilon}
\] and
\[
\begin{vmatrix}
\phi_{d,\epsilon}(V_1^{d,\epsilon}, V_1^{d,\epsilon}) & \phi_{d,\epsilon}(V_0^{d,\epsilon}, V_1^{d,\epsilon}) \\
\phi_{d,\epsilon}(V_1^{d,\epsilon}, V_0^{d,\epsilon}) & \phi_{d,\epsilon}(V_0^{d,\epsilon}, V_0^{d,\epsilon})
\end{vmatrix} \sim 4\alpha_2 \epsilon (p-1) \int_{-1}^{1} W_{d,2}(y)^2 \frac{\rho(y)}{1-y^2} \, dy
\] (163)

as \( \epsilon \to 0 \) uniformly for \( |d| < 1 \). Hence, since \( \alpha_2 > 0 \), (159) follows for \( \epsilon \) small and positive and \( |d| < 1 \), which implies that \( \phi_{d,\epsilon} \) is negative in the plane spanned by \( V_0^{d,\epsilon} \) and \( V_1^{d,\epsilon} \). This concludes the proof of Lemma 4.10. □

**Part 3. End of the proof of Lemma 4.8.** From Lemmas 4.9 and 4.10, we define \( \epsilon_0 = \min(\epsilon_1, \epsilon_2) \in (0, 1) \). We will now prove by contradiction that \( \phi_{d,\epsilon_0} \) is negative on \( \mathcal{H}^d_- \) for all \( |d| < 1 \).

From Lemma 4.10 and (144), for all \( |d| < 1 \) and \( \epsilon \in (0, \epsilon_0] \), we write the definition of \( \mathcal{H}^d_- \) (133) as follows:

\[
\mathcal{H}^d_- = \{ r \in \mathcal{H} \mid \phi_{d,\epsilon}(V^d_\lambda, r) = 0 \text{ for all } \lambda \in \{0, 1\} \}. \quad (164)
\]

We proceed by contradiction and assume that

there is \( r \in \mathcal{H}^d_- \) such that \( \phi_{d,\epsilon}(r, r) < 0 \). (165)

Since the determinant in (163) is not zero, we see from (164) that \( r \notin \text{span}(V_1^{d,\epsilon}, V_0^{d,\epsilon}) \). Therefore, the vector subspace

\[
E_1 = \text{span}(V_1^{d,\epsilon}, V_0^{d,\epsilon}, r)
\]
is of dimension 3. Hence, since the subspace \( E_2 \) (141) is of codimension 2, there exists a non-zero \( u \in E_1 \cap E_2 \).

On the one hand, since \( u \in E_2 \), we have from Lemma 4.9 that

\[
\phi_{d,\epsilon}(u, u) \geq 0. \quad (166)
\]

On the other hand, since \( \phi_{d,\epsilon} \) is negative on \( E_1 \) by (iii) of Lemma 4.10, we must have from (164) and (165),

\[
\phi_{d,\epsilon}(u, u) < 0.
\]

This contradicts (166). Thus, (165) does not hold, and \( \phi_{d,\epsilon} \) is non-negative on \( \mathcal{H}^d_- \). This concludes the proof of Lemma 4.8 and Proposition 4.7. □

**5. Trapping near the set of stationary solutions**

We prove Theorem 3 in this section. Note that in this section, we work in the space \( \mathcal{H} \), which is a natural choice. Indeed, if \( (w, \partial_t w) \in \mathcal{H} \), then the Lyapunov functional \( E(w) \) (15) is well defined, thanks to the Hardy–Sobolev inequality of Lemma 2.2.

We proceed in 3 steps, each of them making a separate subsection.
In Section 5.1, assuming that (18) holds for some $s^* \in \mathbb{R}$, $d^* \in (-1, 1)$, $\omega^* = \pm 1$ and $\epsilon^* > 0$ small enough and independent of $d^*$, we use modulation theory to introduce a parameter $d(s)$ adapted to the linearized operator of Eq. (7) around the stationary solution $\kappa(d, \cdot)$ (see Section 4).

In Section 5.2, under the a priori estimate that $\| (w(s), \partial_s w(s)) - (\kappa(d(s), \cdot), 0) \|_{\mathcal{H}}$ is small, we project the linearized equation of (7) around $\kappa(d(s), \cdot)$ and derive from the energy barrier (17) the smallness of the unstable direction with respect to the stable.

In Section 5.3, we use the two first steps and prove Theorem 3 by showing the convergence of $(w(s), \partial_s w(s))$ to some $\kappa(d_{\infty}, \cdot)$ as $s \to \infty$ in the norm of $\mathcal{H}$.

### 5.1. Modulation theory

In this section, we use modulation theory and introduce a parameter $d(s)$ adapted to the dispersive property of Eq. (7) whenever (18) holds. We claim the following.

**Proposition 5.1** (Modulation of $w$ with respect to $\kappa(d, \cdot)$). There exist $\epsilon_1 > 0$ and $K_1 > 0$ such that if $(w, \partial_s w) \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution to Eq. (7) which satisfies (18) for some $|d^*| < 1$, $\omega^* = \pm 1$ and $\epsilon^* \leq \epsilon_1$, then the following is true:

(i) Choice of the modulation parameter. There exists $d(s) \in C^1([s^*, \infty), \mathbb{R})$ such that for all $s \in [s^*, \infty)$,

$$
\pi_0^d(q(s)) = 0,
$$

where $\pi_0^d$ is defined in (129), $q = (q_1, q_2)$ is defined for all $s \in [s_0, \infty)$ by

$$
\begin{pmatrix}
  w(y, s) \\
  \partial_s w(y, s)
\end{pmatrix} = \begin{pmatrix}
  \kappa(d(s), y) \\
  0
\end{pmatrix} + \begin{pmatrix}
  q_1(y, s) \\
  q_2(y, s)
\end{pmatrix}.
$$

Moreover,

$$
\left| \log\left( \frac{1 + d(s^*)}{1 - d(s^*)} \right) - \log\left( \frac{1 + d^*}{1 - d^*} \right) \right| + \|q(s^*)\|_{\mathcal{H}} \leq K_1 \epsilon^*.
$$

(ii) Equation on $q$. For all $s \in [s^*, \infty)$:

$$
\frac{\partial}{\partial s} \begin{pmatrix}
  q_1 \\
  q_2
\end{pmatrix} = L_d(s) \begin{pmatrix}
  q_1 \\
  q_2
\end{pmatrix} + \begin{pmatrix}
  0 \\
  f_d(s)(q_1)
\end{pmatrix} - d'(s) \begin{pmatrix}
  \partial_d \kappa(d, y) \\
  0
\end{pmatrix},
$$

where

$$
L_d \begin{pmatrix}
  q_1 \\
  q_2
\end{pmatrix} = \left( \mathcal{L} q_1 + \psi(d, \cdot) q_1 - \frac{p+3}{p-1} q_2 - 2y q_2' \right),
$$

$$
f_d(q_1) = |\kappa(d, \cdot) + q_1|^{p-1} (\kappa(d, \cdot) + q_1) - \kappa(d, \cdot)^p - p \kappa(d, \cdot)^{p-1} q_1,
$$

$\mathcal{L}$, $\psi(d, \cdot)$ and $\kappa(d, \cdot)$ are defined respectively in (8) and (41) and (13).
Remark. We recall from (129) that \( \pi_d^0 \) is the projection on \( F_d^0 \) (110), the null eigenspace of \( L_d \) span by \( (\partial_d \kappa(d, y), 0) \) by (110) and (114). In particular, the modulation term (i.e. containing \( d'(s) \)) in (170) is proportional to \( F_d^0 \).

Proof of Proposition 5.1. Up to replacing \( w(y, s) \) by \( -w(y, s) \), we can assume that \( \omega^* = 1 \) in (18).

(i) In (18), we see that there is a parameter \( d^* \in (-1, 1) \) which makes the distance between the solution \((w(s^*), \partial_s w(s^*))\) and a particular element of the family of stationary solutions \( \{ (\kappa(d, y), 0) \mid |d| < 1 \} \) small. Now, we would like to sharpen the decomposition and find for all \( s \in [s^*, \sigma^*] \) for some \( \sigma^* > s^* \) a different parameter \( d(s) \) close to \( d^* \) which not only makes the difference between \((w(s), \partial_s w(s))\) and \( \kappa(d(s), \cdot) \) small, but also satisfies the orthogonality condition (167).

From (129), we see that condition (167) becomes \( \Phi((w(s), \partial_s w(s)), d) = 0 \) where \( \Phi \in C(H \times (-1, 1), \mathbb{R}) \) is defined by

\[
\Phi(v, d) = \phi(v - (\kappa(d, \cdot), 0), W_0^d) \tag{172}
\]

and \( \phi \) and \( W_0^d \) are given in (103) and Lemma 4.4. The implicit function theorem allows us to conclude. Indeed,

- Note first that we have

\[
\Phi((\kappa(d^*, \cdot), 0), d^*) = 0. \tag{173}
\]

- Then, we compute by (172), the expressions of \( \partial_d \kappa(d, y) \) (114) and \( F_0^d \) (110) and the orthogonality relation (118),

\[
D_v \Phi(v, d)(u) = \phi(u, W_0^d) \quad \text{for all } u \in \mathcal{H},
\]

\[
\partial_d \Phi(v, d) = -\phi((\partial_d \kappa(d, \cdot), 0), W_0^d) + \phi(v - (\kappa(d, \cdot), 0), \partial_d W_0^d),
\]

\[
= \frac{2\kappa_0}{(p-1)(1-d^2)} + \phi(v - (\kappa(d, \cdot), 0), \partial_d W_0^d).
\]

Using the continuity of \( \phi \) in \( \mathcal{H} \), the bound (119), and the fact that

\[
\forall d_1, d_2 \in (-1, 1), \quad \| \kappa(d_1, \cdot) - \kappa(d_2, \cdot) \|_{\mathcal{H}_0} \leq C_0 |\theta_1 - \theta_2|, \quad \text{where } \theta_i = \frac{1}{2} \log \left( \frac{1 + d_i}{1 - d_i} \right) \tag{174}
\]

(see below for the proof of (174)), we see that if

\[
\left| \log \left( \frac{1 + d}{1 - d} \right) - \log \left( \frac{1 + d^*}{1 - d^*} \right) \right| + \| v - (\kappa(d^*, \cdot), 0) \|_{\mathcal{H}_0} \leq \epsilon_1
\]

for some \( \epsilon_1 > 0 \) small enough independent of \( d^* \), then we have

\[
\| D_v \Phi(v, d) \| \leq C_0 \quad \text{and} \quad 0 < \frac{1}{C_0(1-d^2)} \leq \partial_d \Phi(v, d) \leq \frac{C_0}{1-d^2}. \tag{175}
\]
Now, if we introduce \( \Psi \in C(\mathcal{H} \times \mathbb{R}, \mathbb{R}) \) defined by

\[
\Psi(v, \theta) = \Phi(v, d) \quad \text{where} \quad d = \tanh \theta,
\]

then, since \( \theta = \frac{1}{2} \log \left( \frac{1+d}{1-d} \right) \) and \( \tanh'(\theta) = 1 - \tanh^2(\theta) \), we see from (173) and (175) that the implicit function theorem applies to \( \Psi \) and we get the existence of \( d(s) \) for all \( s \in [s^*, \sigma^*] \) for some \( \sigma^* \leq \infty \). Assume by contradiction that \( \sigma^* < +\infty \). Applying the implicit function theorem around \( (v, d) = ((w(s_n), \partial_s w(s_n)), d(s_n)) \) where \( s_n = \sigma^* - \frac{1}{n} \), and the uniform continuity of \((w(s), \partial_s w(s))\) from \([\sigma_*, -\eta_0, \sigma_* + \eta_0] \) to \( \mathcal{H} \) for some \( \eta_0 > 0 \), we see that for \( n \) large enough, we can define \( d(s) \) for all \( s \in [s_n, s_n + \epsilon_0] \) for some \( \epsilon_0 > 0 \) independent of \( n \). Therefore, for \( n \) large enough, \( d(s) \) exists beyond \( \sigma^* \), which is a contradiction. Thus, \( \sigma^* = \infty \) and (i) is proved.

It remains to prove (174).

**Proof of (174).**

**Case \( d_1 = 0 \).** Since \( \kappa(d_2, \cdot) = \mathcal{T}_{d_2} \kappa_0 \) by (33), we see from Lemma 2.8 that for all \( d_2 \in (-1, 1) \), \( \| \kappa(d_2, \cdot) \|_{\mathcal{H}_0} \leq \| \kappa_0 \|_{\mathcal{H}_0} \leq C \). Therefore, \( \| \kappa(d_2, \cdot) - \kappa_0 \|_{\mathcal{H}_0} \) is a bounded \( C^1 \) function of \( \theta_2 = \frac{1}{2} \log \left( \frac{1+d_2}{1-d_2} \right) \) which is zero when \( d_2 \) is zero. This directly implies (174).

**Case \( d_1 \neq 0 \).** Using the remark after Lemma 2.6, we see that \( \kappa(d_2, \cdot) - \kappa(d_1, \cdot) = \mathcal{T}_{d_1} \kappa(d_2 * (-d_1)) - \kappa_0 \). Using the continuity estimate of \( \mathcal{T}_{d_1} \) in \( \mathcal{H}_0 \) (see Lemma 2.8) and the case \( d_1 = 0 \), we see that

\[
\| \kappa(d_1, \cdot) - \kappa(d_2, \cdot) \|_{\mathcal{H}_0} \leq C_0 \| \kappa(d_2 * (-d_1), \cdot) - \kappa_0 \|_{\mathcal{H}_0} \leq C_0 |\tilde{\theta}|,
\]

where \( \tilde{\theta} = \frac{1}{2} \log \left( \frac{1+d_2 * (-d_1)}{1-d_2 * (-d_1)} \right) \), or equivalently, \( \tanh \tilde{\theta} = d_2 * (-d_1) \). Since we have from (32)

\[
d_2 * (-d_1) = \frac{d_2 - d_1}{1 - d_2 d_1} = \frac{\tanh \theta_2 - \tan \theta_1}{1 - \tan \theta_1 \tan \theta_2} = \tanh(\theta_2 - \theta_1),
\]

we see that \( \tilde{\theta} = \theta_2 - \theta_1 \), which concludes the proof of (174) and of Proposition 5.1(i).

(ii) is a direct consequence of Eq. (7) satisfied by \( w \) put in vectorial form:

\[
\begin{align*}
\partial_s w &= v, \\
\partial_s v &= \mathcal{L} w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} v - 2y \partial_s v
\end{align*}
\]

and the fact that \((\kappa(d, \cdot), 0)\) is a stationary solution of (176)–(177), that is \( \kappa(d, \cdot) \) is a solution of

\[
\begin{align*}
\mathcal{L} \kappa(d, \cdot) - \frac{2(p+1)}{(p-1)^2} \kappa(d, \cdot) + |\kappa(d, \cdot)|^{p-1} \kappa(d, \cdot) &= 0
\end{align*}
\]

(see Proposition 1).

Indeed, since we have from (168), the definition of \( \mathcal{L} \) (8) and \( f_d(q_1) \) (171):
\[ w(y, s) = q_1(y, s) + \kappa(d(s), y), \]
\[ \mathcal{L}w(y, s) = \mathcal{L}q_1(y, s) + \mathcal{L} \kappa(d(s), y), \]
\[ |w|^{p-1}w(y, s) = f_d(q_1) + \kappa(d(s), y)^p + pk(d, y)^{p-1}q_1(y, s), \]

and from (176) and (168), \( v = \partial_t w = q_2 \), we see that Eq. (170) follows immediately from (176)–(178). This concludes the proof of Proposition 5.1. \( \square \)

5.2. **Projection on the eigenspaces of the operator** \( L_d \)

Given \( s \geq s^* \) and following the previous section, we make in this subsection the following a priori estimate:

\[ \|q(s)\|_{\mathcal{H}} \leq \epsilon \]

for some \( \epsilon > 0 \). From (167), we will expand \( q \) according to the spectrum of the linear operator \( L_d \) as in (130):

\[ q(y, s) = \alpha_1(s) F_1^{d(s)}(y) + q_-(y, s), \]

where

\[ \alpha_1(s) = \pi_1^{d(s)}(q), \quad \alpha_0(s) = \pi_0^{d(s)}(q) = 0, \quad \alpha_-(s) = \sqrt{\varphi_d(q_-, q_-)} \]  

and

\[ q_- = \begin{pmatrix} q_{-1} \\ q_{-2} \end{pmatrix} = \pi_-^d(q) = \pi_-^d \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \]

From (180) and Proposition 4.7, we see that for all \( s \geq s_0 \),

\[ \frac{1}{C_0} \alpha_-(s) \leq \|q_-(s)\|_{\mathcal{H}} \leq C_0 \alpha_-(s), \]

\[ \frac{1}{C_0} \left( |\alpha_1(s)| + \alpha_-(s) \right) \leq \|q(s)\|_{\mathcal{H}} \leq C_0 \left( |\alpha_1(s)| + \alpha_-(s) \right) \]  

for some \( C_0 > 0 \). In the following proposition, we derive from (170) differential inequalities satisfied by \( \alpha_1(s) \), \( \alpha_-(s) \) and \( d(s) \).

**Proposition 5.2.** There exists \( \epsilon_2 > 0 \) such that if \( w \) is a solution to Eq. (7) satisfying (167) and (179) at some time \( s \) for some \( \epsilon \leq \epsilon_2 \), where \( q \) is defined in (168), then:

(i) **(Control of the modulation parameter)**

\[ |d'| \leq C_0 \left( 1 - d^2 \right) \left( \alpha_1^2 + \alpha_-^2 \right). \]  

\[ \]
(ii) (Projection of Eq. (170) on the different eigenspaces of $L_d$)

\[ |\alpha'_1 - \alpha_1| \leq C_0(\alpha^2_1 + \alpha^2), \quad (185) \]

\[ \left( R_- + \frac{1}{2} \alpha_-^2 \right)' \leq -\frac{4}{p-1} \int_{-1}^{1} q_-' - \frac{\rho}{1 - y^2} dy + C_0(\alpha^2_1 + \alpha^2)^{3/2} \]

(186)

for some $R_-(s)$ satisfying

\[ |R_-(s)| \leq C_0(\alpha^2_1 + \alpha^2)^{\frac{1+p}{2}}, \text{ where } \bar{p} = \min(p, 2) > 1. \]

(187)

(iii) (Additional relation)

\[ \frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho \leq -\frac{4}{5} \alpha_-^2 + C_0 \int_{-1}^{1} q_-' - \frac{\rho}{1 - y^2} + C_0 \alpha_1^2. \]

(188)

(iv) (Energy barrier) If moreover (17) holds, then

\[ \alpha_1(s) \leq C_0 \alpha_-(s). \]

(189)

**Remark.** Here, (186) and (188) are coming from the relations we use in [17] to bound uniformly $(w(s), \partial_s w(s))$ in $H^1 \times L^2(-1, 1)$. Identities (186) and (188) together will be fundamental to control the dynamics of the infinite-dimensional part $q_-$ of the solution, and allow us thus to overcome the difficulty coming from the non-self-adjoint character of the linear operator $L_d$. Such a use of conservation laws to control the dynamics is in the same spirit as the case of NLS (Viriel identity and the mass ejection law; see Merle and Raphaë [15,16].)

**Proof of Proposition 5.2.** Before the proof, let us give the following nonlinear estimate.

**Claim 5.3 (Nonlinear estimates).** For all $y \in (-1, 1)$,

\[ |f_d(s)(q_1(y, s))| \leq mM(\kappa(d(s), y)^{p-2}|q_1(y, s)|^2, C_0|q_1(y, s)|^p), \quad (190) \]

\[ |\mathcal{F}_d(s)(q_1(y, s))| \leq mM(\kappa(d(s), y)^{p-2}|q_1(y, s)|^3, C_0|q_1(y, s)|^{p+1}), \quad (191) \]

where $mM = \min$ if $1 < p < 2$ and $mM = \max$ if $p \geq 2$, and

\[ \mathcal{F}_d(q_1) = \int_{0}^{q_1} f_d(q') dq' \]

\[ = \frac{|\kappa(d, \cdot) + q_1|^{p+1}}{p+1} - \frac{\kappa(d, \cdot)^{p+1}}{p+1} - \kappa(d, \cdot) p \frac{p}{2} \kappa(d, \cdot)^{p-1} q_1^2. \]

(192)
Proof. Introducing $\xi = q_1 / \kappa(d(s), y)$ and considering the cases where $|\xi| < 1$ and $|\xi| \geq 1$, we directly get (i). Since (ii) follows from (i) by integration, this concludes the proof of Claim 5.3. □

(i), (ii). We proceed in 2 steps:

- In Step 1, we project Eq. (170) with the projector $\pi^d_\lambda$ (129) for $\lambda = 0$ and $\lambda = 1$ and derive the smallness condition on $d' (184)$ and the equation satisfied by $\alpha_1 (185)$.
- In Step 2, we write an equation satisfied by $(q_-, 1, q_-, 2)$ which is the difficult part in this non-self-adjoint framework. We claim that (186) follows from the existence of the Lyapunov functional $E(w) (15)$ for Eq. (7). Here, the Lyapunov functional structure will be revealed by the quadratic form $\varphi_d (134)$.

Step 1. Projection of Eq. (170) on the modes $\lambda = 0$ and $\lambda = 1$. Projecting Eq. (170) with the projector $\pi^d_\lambda$ (129) for $\lambda = 0$ and $\lambda = 1$, we write

$$
\pi^d_\lambda (\partial_s q) = \pi^d_\lambda (L_d q) + \pi^d_\lambda \left( \frac{0}{f_d(q_1)} \right) - d'(s) \pi^d_\lambda \left( \frac{\partial_d \kappa(d, y)}{0} \right).
$$

(193)

- Since $\alpha_\lambda(s) = \pi^d_\lambda (q) = \phi(W^d_\lambda, q)$ by (181) and the definition of $\pi^d_\lambda (129)$, we write

$$
\alpha_\lambda'(s) = \pi^d_\lambda (\partial_s q) + d'(s) \phi(\partial_d W^d_\lambda, q).
$$

Using (119) and (183), we get

$$
|\pi^d_\lambda (\partial_s q) - \alpha_\lambda'(s)| \leq C_0 \frac{1}{d^2} |d'|(|\alpha_1| + \alpha_-).
$$

(194)

- Using (i) of Lemma 4.4, the definition of $\pi^d_\lambda (129)$, the duality relation (106) and (181), we write

$$
\pi^d_\lambda (L_d(q)) = \phi(W^d_\lambda, L_d(q)) = \phi(L^*_d(W^d_\lambda), q) = \lambda \phi(W^d_\lambda, q) = \lambda \pi^d_\lambda (q) = \lambda \alpha_\lambda(s).
$$

(195)

- Using (46), the definition of $W^d_{\lambda, 2}$ (116) and (121), we have

$$
\forall (d, y) \in (-1, 1)^2, \quad |W^d_{\lambda, 2}(y)| \leq C \kappa(d, y).
$$

(196)

Therefore, using the definitions of $\pi^d_\lambda (129)$ and $\phi (102)$, and Claim 5.3, we see that

$$
\left| \pi^d_\lambda \left( \frac{0}{f_d(q_1)} \right) \right| \leq C \int_{-1}^{1} \kappa(d, y) |f_d(q_1)| \rho(y) dy
$$

(197)

$$
\leq C_0 \int_{-1}^{1} \kappa(d, y)^{p-1} q_1(y, s)^2 \rho dy + C_0 \delta_{p \geq 2} \int_{-1}^{1} \kappa(d, y) |q_1(y, s)|^p \rho dy
$$

$$
\leq C_0 \|q_1\|_{L^{p+1}}^{2} \|\kappa(d, y)\|_{L^{p+1}}^{p-1} + C_0 \delta_{p \geq 2} \|q_1\|_{L^{p+1}}^{p} \|\kappa(d, y)\|_{L^{p+1}},
$$

(198)
where $\delta_{\{p \geq 2\}}$ is 0 if $1 < p < 2$ and 1 otherwise. Therefore, using (49), (197), (198), Lemma 2.2, (179) and (183), we get

$$\left\| \pi_d^\lambda \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix} \right\| \leq C \int_{-1}^{1} \kappa(d, y) |f_d(q_1)| \rho(y) \, dy \leq C_0 (\alpha_1(s)^2 + \alpha_-(s)^2).$$  \hspace{1cm} (199)

- Using (114), (110), (131) and (121), we write

$$\pi_d^\lambda \begin{pmatrix} \partial_d \kappa(d, y) \\ 0 \end{pmatrix} = -\frac{2\kappa_0}{(p-1)(1-d^2)} \pi_d^\lambda (F_0^d) = -\frac{2\kappa_0}{(p-1)(1-d^2)} \delta_{\lambda, 0}. \hspace{1cm} (200)$$

- Using (193), (194), (195), (199), (200) and the fact that $\alpha_0 \equiv \alpha_0' \equiv 0$ by (181), we get for $\lambda = 0, 1$:

$$\frac{2\kappa_0}{(p-1)(1-d^2)} |d'| \leq \frac{C_0}{d^2} |d'| (|\alpha_1| + \alpha_-) + C_0 (\alpha_1^2 + \alpha_-^2),$$

$$|\alpha'_1(s) - \alpha_1(s)| \leq \frac{C_0}{d^2} |d'| (|\alpha_1| + \alpha_-) + C_0 (\alpha_1^2 + \alpha_-^2).$$

Using the smallness condition (179) and (183), we obtain (184) and (185) for $\epsilon$ small enough.

**Step 2.** Differential inequality on $\alpha_-$. In the following claim, we project Eq. (170) on the negative modes, which gives a partial differential inequality satisfied by $q_-$:

**Claim 5.4 (Preliminary estimates).** There exists $\epsilon_3 > 0$ such that if $\epsilon \leq \epsilon_3$ in the hypotheses of Proposition 5.2, then

$$\left\| \partial_sq_- - L_d(q_-) - \pi_d^d \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 (\alpha_1^2 + \alpha_-^2)^{3/2}, \hspace{1cm} (201)$$

$$\left\| \varphi_d \left( q_- - \pi_d^d \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix} \right) - \int_{-1}^{1} q_2 f_d(q_1) \rho \, dy \right\| \leq C_0 (\alpha_1^2 + \alpha_-^2)^{3/2}, \hspace{1cm} (202)$$

$$\left\| \int_{-1}^{1} q_2 f_d(q_1) \rho \, dy - \frac{d}{ds} \int_{-1}^{1} \mathcal{F}_d(q_1) \rho \, dy \right\| \leq C_0 (\alpha_1^2 + \alpha_-^2)^2, \hspace{1cm} (203)$$

where $\mathcal{F}_d(q_1)$ is defined in (192).

**Remark.** Note that the term in (203) cannot be controlled directly and has to be seen as a time derivative.

Assuming now Claim 5.4, we are able to conclude the proof of the differential inequality (186) satisfied by $\alpha_-$. 

Proof of (186) assuming Claim 5.4. In fact, the whole proof is based on the fact that the derivative of \( \alpha^2 \) is related to the quadratic form \( \varphi_d(q_-, L_d(q_-)) \) defined in (134), which inherits the properties of the Lyapunov functional defined in (15) (and give an almost self-adjoint behavior). Note from the definition we took for \( \alpha_- \) (181) that

\[
\alpha_- (s)^2 = \varphi_d(q_-(s), q_-(s)).
\]

Using the definition (134) of \( \varphi_d \), we have by differentiation

\[
\alpha_- \varphi_-' = \varphi_d(q_-, \partial_s q_-) - \frac{1}{2} d'(s) \int_{-1}^{1} \partial_d \psi (d, y) q_{2,-1}^{2} \rho. \tag{204}
\]

Using the Hölder inequality, the Hardy–Sobolev estimate of Lemma 2.2 and (183), we write

\[
\left| \int_{-1}^{1} \partial_d \psi (d, y) q_{2,-1}^{2} \rho \right| \leq \left\| \partial_d \psi (d, y) \right\|_{L_p}^{p+1} \| q_{2,-1} \|_{L_p}^{2} \leq C_0 \left| \partial_d \psi (d, y) \right|_{L_p}^{p+1} \alpha_- (s)^2. \tag{205}
\]

Since \( |\partial_d \psi (d, y)| \leq C/(1 + dy)^2 \) for all \((d, y) \in (-1, 1)^2 \) by (41), using Claim 4.3, we see that

\[
\left| \partial_d \psi (d, y) \right|_{L_p}^{p+1} \leq C/(1 - d^2).
\]

Therefore, using (204), (205), and the bound (184) on \(|d'(s)|\), we get

\[
|\alpha_- \varphi_- - \varphi_d(q_-, \partial_s q_-)| \leq C_0|d'| \frac{\alpha_-^2}{1 - d^2} \leq C_0(\alpha_1^2 + \alpha_-^2)^2. \tag{206}
\]

From (206), the continuity of \( \varphi_d \) (138), Claim 5.4, (183), we write

\[
\left| \alpha_- \varphi_- - \varphi_d(q_-, L_d(q_-)) \right| - \frac{d}{ds} \int_{-1}^{1} F_d(q_1) \rho \, dy \leq C_0(\alpha_1^2 + \alpha_-^2)^{3/2} + \left| \varphi_d\left(q_-, \partial_s q_- - L_d(q_-) - \pi_d\left(\frac{0}{f_d(q_1)}\right)\right) \right| \leq C_0(\alpha_1^2 + \alpha_-^2)^{3/2} + \| q_- \|_{H}(\alpha_1^2 + \alpha_-^2)^{3/2} \leq C_0(\alpha_1^2 + \alpha_-^2)^{3/2}. \tag{207}
\]

On the one hand, using the expressions of \( L_d \) (171) and \( \varphi_d \) (135), we have

\[
\varphi_d(q_-, L_d(q_-)) = \varphi_d\left(\left(q_{2,-1,q} + \psi (d, y) q_{2,-1} - \frac{p+3}{p-1} q_{2,-2} - 2 y q_-\right), \left(q_{2,-1}\right)\right) = - \int_{-1}^{1} q_{2,-1}(Lq_{2,-1} + \psi (d, y) q_{2,-1}) \rho \, dy
\]
\[
+ \int_{-1}^{1} \left( \mathcal{L}q_{-1} + \psi(d,y)q_{-1} - \frac{p+3}{p-1} q_{-2} - 2yq'_{-2} \right) q_{-2} \rho(y) dy
\]

\[
= -\frac{p+3}{p-1} \int_{-1}^{1} q_{-2}^2 \rho dy - \int_{-1}^{1} y(q_{-2}^2)' \rho dy
\]

\[
= -\frac{p+3}{p-1} \int_{-1}^{1} q_{-2}^2 \rho dy + \int_{-1}^{1} q_{-2}^2 (\rho - y\rho') dy
\]

\[
= -\frac{4}{p-1} \left[ \int_{-1}^{1} q_{-2}^2 \rho dy + \int_{-1}^{1} \frac{y^2 \rho}{1-y^2} dy \right]
\]

\[
= -\frac{4}{p-1} \int_{-1}^{1} q_{-2}^2 \frac{\rho}{1-y^2} dy.
\]  

(208)

Using (207) and (208), we see that estimate (186) holds with

\[
R_{-}(s) = -\int_{-1}^{1} \mathcal{F}_d(q_1) \rho dy.
\]  

(209)

Using Claim 5.3(ii), Lemma 2.2 and condition (179) (considering first the case \( p \geq 2 \) and then the case \( 1 < p < 2 \), we see that (187) holds. Remains to prove Claim 5.4 to conclude the proof of (i), (ii) of Proposition 5.2.

**Proof of Claim 5.4.** *Proof of (201).* We first project Eq. (170) using the negative projector \( \pi_d^- \) introduced in Definition 4.6:

\[
\pi_d^-(\partial_s q) = \pi_d^-(L_d q) + \pi_d^-(\begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix}) - d'(s) \pi_d^-(\partial_d \kappa(d, y)).
\]  

(210)

• We will use the notation (182) here. Differentiating (180) and using the expansion (130) with \( \partial_s q \), we write

\[
\partial_s q(y,s) = \alpha'_1(s) F_1^d(y) + \alpha_1(s)d'(s)\partial_d F_1^d(y) + \partial_s q_-(y,s),
\]  

(211)

\[
\partial_s q(y,s) = \pi_1^d(\partial_s q) F_1^d(y) + \pi_0^d(\partial_s q) F_0^d(y) + \pi_0^d(\partial_s q).
\]  

(212)

Making the difference between (211) and (212) and using (111), we get

\[
\|\pi_d^d(\partial_s q) - \partial_s q_-(y,s)\|_{\mathcal{H}} \leq C_0 \left( |\pi_1^d(\partial_s q) - \alpha'_1(s)| + |\pi_0^d(\partial_s q)| + \frac{|\alpha_1 d'(s)|}{1 - d^2} \right).
\]
Using (194), (167) and (184), we obtain
\[
\| \pi d (\partial_s q) - \partial_s q_-(y, s) \|_{\mathcal{H}} \leq C_0 (\alpha_1^2 + \alpha_-^2)^{3/2}. \tag{213}
\]

• Applying the operator \( L_d \) to (180) and using the fact that \( L_d F_1^d = F_1^d \) (see Lemma 4.2), we obtain
\[
L_d q = \alpha_1 (s) F_1^d + L_d (q_-). \tag{214}
\]

Since \( \pi d (F_1^d) = 0 \) and \( \pi d (L_d (q_-)) = L_d (q_-) \) (see the remark after Definition 4.6 and note in particular that \( L_d (q_-) \in \mathcal{H}_d^- \) because \( q_- \in \mathcal{H}_d^- \)), we get from (214)
\[
\pi d (L_d (q)) = L_d (q_-). \tag{215}
\]

• Using (114), (110) and the remark after Definition 4.6, we write
\[
\pi_-(d) \left( \begin{array}{c}
\partial d \kappa (d, y) \\
0
\end{array} \right) = - \frac{2 \kappa_0}{p - 1} (1 - d^2)^{-1} \pi d (F_0^d) = 0. \tag{216}
\]

Using (210), (213), (215) and (216), we write
\[
\| \partial_s q_- - L_d (q_-) - \pi_-(d) \begin{pmatrix} 0 \\ f_d (q_1) \end{pmatrix} \| \leq C_0 (\alpha_1^2 + \alpha_-^2)^{3/2}. \]

This concludes the proof of (201).

**Proof of (202).** Recall from (180) and (130) that we have
\[
q (y, s) = \alpha_1 (s) F_1^d (y) + q_- (y, s), \tag{217}
\]
\[
\begin{pmatrix} 0 \\ f_d (q_1) \end{pmatrix} = \beta_1 (s) F_1^d (y) + \beta_0 (s) F_0^d (y) + \pi_-(d) \begin{pmatrix} 0 \\ f_d (q_1) \end{pmatrix}, \tag{218}
\]

where

\[
\beta_\lambda (s) = \pi_\lambda \left( \begin{array}{c}
0 \\
f_d (q_1)
\end{array} \right).
\]

Note from the definition (134) and the bilinearity of \( \varphi_d \), the bound on the norm of \( \mathcal{F}_\lambda(d) \) (111), (138) and (183) that

\[
\left| \varphi_d \left( q_-, \pi_-(d) \begin{pmatrix} 0 \\ f_d (q_1) \end{pmatrix} \right) - \int_{-1}^1 q_2 f_d (q_1) \rho \, dy \right|
\]
\[
= \left| \varphi_d \left( q_-, \pi_-(d) \begin{pmatrix} 0 \\ f_d (q_1) \end{pmatrix} \right) - \varphi_d \left( q, \begin{pmatrix} 0 \\ f_d (q_1) \end{pmatrix} \right) \right|
\]
\[
\leq C_0 (|\alpha_1| + |\alpha_-|) (|\beta_1| + |\beta_0|) + |\alpha_1| \left| \varphi_d \left( F_1^d, \begin{pmatrix} 0 \\ f_d (q_1) \end{pmatrix} \right) \right|.
\]
Since we have from the expression (134) of $\varphi_d$, the fact that $|F_{1,2}(y)| \leq C_\kappa(d, y)$ and (199),

$$\varphi_d \left( F_1^d \cdot \left( \begin{array}{c} 0 \\ f_d(q_1) \end{array} \right) \right) \leq \int_{-1}^{1} F_{1,2}(y) f_d(q_1) \rho(y) dy \leq C_0 (\alpha_1^2 + \alpha_2^2), \quad (219)$$

$$|\beta_1(s)| + |\beta_0(s)| \leq C_0 \int_{-1}^{1} \kappa(d, y) |f_d(q_1)| \rho dy \leq C_0 (\alpha_1^2 + \alpha_2^2), \quad (220)$$

this gives (202).

**Proof of (203).** Since $q_2 = \partial_s q_1 + d' \partial_d \kappa(d, y)$ by (168), we use (192) to write

$$\int_{-1}^{1} q_2 f_d(q_1) \rho dy = \int_{-1}^{1} \partial_s q_1 f_d(q_1) \rho dy + \int_{-1}^{1} d'(s) \int_{-1}^{1} (\partial_d \kappa(d, y) f_d(q_1) - \partial_d F_d(q_1)) \rho dy \rho dy$$

$$= \frac{d}{ds} \int_{-1}^{1} F_d(q_1) \rho dy + d'(s) \int_{-1}^{1} (\partial_d \kappa(d, y) f_d(q_1) - \partial_d F_d(q_1)) \rho dy$$

$$= \frac{d}{ds} \int_{-1}^{1} F_d(q_1) \rho dy + d'(s) \frac{p(p-1)}{2} \int_{-1}^{1} \partial_d \kappa(d, y) \kappa(d, y) p^{-2} q_1(y, s)^2 \rho dy.$$

(221)

Since we have $\|\partial_d \kappa(d, y) \kappa(d, y) p^{-2}\|_{L_p^{p+1}} \leq C_0/(1 - d^2)$, from the definitions of $\partial_d \kappa(d, y)$ (114), $F_0^d$ (110) and Claim 4.3, we use the Hölder inequality and the Hardy–Sobolev inequality of Lemma 2.2 to derive that

$$\int_{-1}^{1} \int_{-1}^{1} \partial_d \kappa(d, y) \kappa(d, y) p^{-2} q_1(y, s)^2 \rho dy \leq \frac{C_0}{1 - d^2} \|q_1\|_{L_p^{p+1}}^2 \leq \frac{C_0}{1 - d^2} \|q(s)\|_{\mathcal{H}}^2. \quad (222)$$

Using (183) and (184), we see that (221) and (222) give (203). This concludes the proof of Claim 5.4 as well as (i), (ii) of Proposition 5.2.  

(iii) This inequality is a consequence of the coercivity of the quadratic form $\varphi_d$ on the space $\mathcal{H}^d$ stated in Proposition 4.7.  

From Eq. (170) and the definition of $L_d$ (171), we write

$$\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho = \int_{-1}^{1} q_2 \partial_s q_1 \rho + \int_{-1}^{1} q_1 \partial_s q_2 \rho.$$
\[ = \int_{-1}^{1} q_2^2 \rho - d'(s) \int_{-1}^{1} q_2 \partial d \kappa(d, y) \rho \\
+ \int_{-1}^{1} q_1 \left( \mathcal{L} q_1 + \psi(d, y) q_1 - \frac{p + 3}{p - 1} q_2 - 2y \partial y q_2 + f_d(q_1) \right) \rho \]  
\quad \text{(223)}

- First, note from (183) that
\[ \int_{-1}^{1} q_2^2 \rho + \int_{-1}^{1} (\partial y q_1)^2 (1 - y^2) \rho + \int_{-1}^{1} q_2^2 \rho \leq C_0(\alpha_1^2 + \alpha_-^2). \]  
\quad \text{(224)}

- Using (180), the Hardy estimate (22) and the bound (111), we write
\[ \int_{-1}^{1} q_2^2 \frac{\rho}{1 - y^2} \leq 2 \int_{-1}^{1} q_2^{-2} \frac{\rho}{1 - y^2} + 2\alpha_1^2 \int_{-1}^{1} (\mathcal{F}_d^{1,1})^2 \frac{\rho}{1 - y^2} \leq 2 \int_{-1}^{1} q_2^{-2} \frac{\rho}{1 - y^2} + C_0 \alpha_1^2. \]  
\quad \text{(225)}

- From the expression of $\varphi_d$ (135), (180), the definition of $\alpha_-$ (181), the continuity estimate (138), the bound (111) on $\mathcal{F}_d^{1,1}$ and (183), we write
\[ \int_{-1}^{1} q_1 \left( \mathcal{L} q_1 + \psi(d, y) q_1 \right) \rho \\
= -\varphi_d \left( \begin{pmatrix} q_1 \\ 0 \end{pmatrix}, \begin{pmatrix} q_1 \\ 0 \end{pmatrix} \right) \\
= -\varphi_d \left( \begin{pmatrix} q_{-1} \\ 0 \end{pmatrix}, \begin{pmatrix} q_{-1} \\ 0 \end{pmatrix} \right) - \alpha_1^2 \varphi_d \left( \begin{pmatrix} \mathcal{F}_d^{1,1} \\ 0 \end{pmatrix}, \begin{pmatrix} \mathcal{F}_d^{1,1} \\ 0 \end{pmatrix} \right) - \alpha_1 \varphi_d \left( \begin{pmatrix} \mathcal{F}_d^{1,1} \\ 0 \end{pmatrix}, \begin{pmatrix} q_{-1} \\ 0 \end{pmatrix} \right) \\
\leq -\alpha_-^2 + \int_{-1}^{1} q_2^{-2} \frac{\rho}{1 - y^2} + C_0(\alpha_1^2 + |\alpha_1|\alpha_-) \\
\leq -\frac{9}{10} \alpha_-^2(s)^2 + \int_{-1}^{1} q_2^{-2} \frac{\rho}{1 - y^2} + C_0 \alpha_1^2. \]

- Since $\|\partial d \kappa(d, y)\|_{L^2_\rho} \leq C_0 / (1 - d^2)$ from the definition of $\partial d \kappa(d, y)$ and Claim 4.3, we use the Cauchy–Schwarz inequality, (184), (224) and (179) to write for $\epsilon$ small enough,
\[ \left| d'(s) \int_{-1}^{1} q_2 \partial d \kappa(d, y) \rho \, dy \right| \leq C|d'(s)| \left( \int_{-1}^{1} q_2^2 \rho \right)^{1/2} \|\partial d \kappa(d)\|_{L^2_\rho} \\
\leq C_0(\alpha_1^2 + \alpha_-^2)^{3/2} \leq \frac{1}{100} (\alpha_1^2 + \alpha_-^2). \]  
\quad \text{(226)}
• Using integration by parts, the fact that $|y \partial_y \rho(y)| \leq C P(y)$, the Cauchy–Schwarz inequality, the Hardy–Sobolev estimate (21), (224) and (225), we write

$$
\left| - \frac{p + 3}{p - 1} \int_{-1}^{1} q_1 q_2 \rho - 2 \int_{-1}^{1} q_1 y \partial_y q_2 \rho \right|
$$

$$
= \left| 2 \int_{-1}^{1} q_2 \partial_y q_1 y \rho + \left( 2 - \frac{p + 3}{p - 1} \right) \int_{-1}^{1} q_2 q_1 \rho + 2 \int_{-1}^{1} q_2 q_1 y \partial_y \rho \right|
$$

$$
\leq C \int_{-1}^{1} \left( |q_2||\partial_y q_1| \rho + |q_2||q_1| \left( \frac{\rho}{1 - y^2} \right) \right)
$$

$$
\leq C \left( \int_{-1}^{1} |q_2|^2 \left( \frac{\rho}{1 - y^2} \right) \right)^{1/2} \left[ \int_{-1}^{1} (\partial_y q_1)^2 (1 - y^2) \rho + \int_{-1}^{1} q_1^2 \left( \frac{\rho}{1 - y^2} \right) \right]^{1/2}
$$

$$
\leq C_0 (\alpha_1^2 + \alpha_2^2)^{1/2} \left( \int_{-1}^{1} q_2 (1 - y^2) \rho + C_0 \alpha_1^2 \right) \leq 100 C_0^2 \int_{-1}^{1} q_2 (1 - y^2) \rho + C_0 \alpha_2^2 + \frac{\alpha_1^2}{100}.
$$

• Using (49), Claim 5.3, the Hölder inequality and Lemma 2.2, (183) and (179), we write for $\epsilon$ small enough (note that $\overline{p} + 1 > 2$ and use (179)),

$$
\int_{-1}^{1} q_1 f_d(q_1) \rho \leq C_0 \delta_{(p \geq 2)} \int_{-1}^{1} (\kappa(d, y))^{p-2} |q_1|^3 \rho + C_0 \int_{-1}^{1} |q_1|^{p+1} \rho
$$

$$
\leq C_0 \delta_{(p \geq 2)} \|\kappa(d, y)\|_{L_{p+1}}^{p-2} \|q_1\|_{L_{p+1}} + C_0 \|q_1\|_{L_{p+1}}^{p+1}
$$

$$
\leq C_0 \|q\|_{\mathcal{H}}^{\overline{p}+1} \leq \frac{1}{100} (\alpha_1^2 + \alpha_2^2). \tag{227}
$$

Collecting (223)–(227) concludes the proof of (iii) of Proposition 5.2.

(iv) Using the definition of $q(y, s)$ (168), we can make an expansion of $E(w(s))$ (15) for $q \to 0$ in $\mathcal{H}$ and get after from straightforward computations

$$
E(w(s)) = E(\kappa(d, \cdot)) + \frac{1}{2} \varphi_d(q, q) - \int_{-1}^{1} \mathcal{F}_d(q_1) \rho dy, \tag{228}
$$

where $\varphi_d$ and $\mathcal{F}_d(q_1)$ are defined in (134) and (192). Note in particular that there is no linear term, since $\kappa(d, \cdot)$ is a stationary solution to (7), hence, a critical point of $E(w(s))$. Moreover, as
we announced right after (134), the second variation of $E(w(s))$ around $\kappa(d, \cdot)$ is given by $\varphi_d$. Since we have (209), (187), (179) and (183)
\[
\left| \int_{-1}^{1} \mathcal{F}_d(q) \rho \, dy \right| \leq C \| q(s) \|_{\mathcal{H}^{\tilde{p} + 1}} \leq C e^{\tilde{p} - 1} (\alpha_1^2 + \alpha_-^2),
\] (229)
where $\tilde{p} = \min(p, 2)$, we claim that the conclusion follows from the fact that
\[
\varphi_d(q, q) \leq C_0 \alpha_2^2 - C_1 \alpha_1^2
\] (230)
for some $C_1 > 0$. Indeed, from (17), (228), (230) and (229), we see that taking $\epsilon$ small enough so that $C \epsilon \bar{p} - 1 \leq C_1$, we get
\[
0 \leq E(w(s)) - E(\kappa(d, \cdot)) \leq \left( \frac{C_0}{2} + \frac{C_1}{4} \right) \alpha_2^2 - \frac{C_1}{4} \alpha_1^2,
\] which yields (189). It remains to prove (230).

Proof of (230). Since $L_d(F_1^d) = F_1^d$ by Lemma 4.2, calculation (208) holds with $q_-$ replaced by $F_1^d$, and we get from Claim 4.3 for some $C_1 > 0$,
\[
\varphi_d(F_1^d, F_1^d) = -\frac{4}{p - 1} \int_{-1}^{1} (F_1^d)^2 \frac{\rho}{1 - y^2} \, dy \leq -2C_1. \tag{231}
\]
Since we have from the decomposition (180), the definition of $\alpha_-$ (181), the continuity of $\varphi_d$ (138), the bound on $F_1^d$ (111), (183) and (231),
\[
\varphi_d(q, q) = \varphi_d(q_-, q_-) + 2\alpha_1 \varphi_d(F_1^d, q_-) + \alpha_1^2 \varphi_d(F_1^d, F_1^d)
\] \[
\leq \alpha_2^2 + \frac{C_0^2}{C_1} \alpha_1^2 + C_1 \alpha_2^2 - 2C_1 \alpha_1^2, \tag{232}
\]
this yields (230) and concludes the proof of Proposition 5.2. \square

5.3. Exponential decay of the different components

We prove Theorem 3 in this subsection. Let us first introduce a more adapted notation and rewrite Proposition 5.2.

If we introduce
\[
\theta(s) = \frac{1}{2} \log \left( \frac{1 + d(s)}{1 - d(s)} \right), \quad a(s) = \alpha_1(s)^2 \quad \text{and} \quad b(s) = \alpha_-(s)^2 + 2R_-(s) \tag{234}
\]
(note that $d(s) = \tanh(\theta(s))$), then we see from (187), and (183) that if (179) holds, then $|b - \alpha_-^2| \leq C_0 e^{\tilde{p} - 1} (\alpha_1^2 + \alpha_-^2)$, hence
\[
\frac{99}{100} \alpha_-^2 - \frac{1}{100} a \leq b \leq \frac{101}{100} \alpha_-^2 + \frac{1}{100} a \tag{235}
\]
for $\epsilon$ small enough. Therefore, using Proposition 5.2, estimate (179), (183) and the fact that $\theta'(s) = \frac{d'(s)}{1-d(s)^2}$, we derive the following.

**Corollary 5.5** (Relations between $a$, $b$, $\theta$ and $\int_{-1}^{1} q_1 q_2 \rho$). There exist positive $\epsilon_4$, $K_4$ and $K_5$ such that if $w$ is a solution to Eq. (7) such that (167) and (179) hold at some time $s$ for some $\epsilon \leq \epsilon_4$, where $q$ is defined in (168), then using the notation (234), we have:

(i) *(Size of the solution)*

$$\frac{1}{K_4} (a(s) + b(s)) \leq \|q(s)\|_\mathcal{H}^2 \leq K_4 (a(s) + b(s)) \leq K_4^2 \epsilon^2, \quad (236)$$

$$|\theta'(s)| \leq K_4 (a(s) + b(s)) \leq K_4^2 \|q(s)\|_\mathcal{H}^2, \quad (237)$$

$$\left| \int_{-1}^{1} q_1 q_2 \rho \right| \leq K_4 (a(s) + b(s)) \quad (238)$$

and (235) holds.

(ii) *(Equations)*

$$\frac{3}{2} a - K_4 \epsilon b \leq a' \leq \frac{5}{2} a + K_4 \epsilon b, \quad (239)$$

$$b' \leq -\frac{8}{p - 1} \int_{-1}^{1} q_2^2 \frac{\rho}{1 - y^2} + K_4 \epsilon (a + b), \quad (240)$$

$$\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho \leq -\frac{3}{5} b + K_4 \int_{-1}^{1} q_2^2 \frac{\rho}{1 - y^2} + K_4 a. \quad (241)$$

(iii) *(Energy barrier)* If (17) holds, then

$$a(s) \leq K_5 b(s). \quad (242)$$

At this level, we still do not have exponential decay of $a$ and $b$. However, with this corollary and the following analysis, we are ready to prove Theorem 3.

**Proof of Theorem 3.** Consider $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ a solution of Eq. (7) such that (17) and (18) hold for some $d^* \in (-1, 1)$, $\omega^* = \pm 1$ and $\epsilon^* \in (0, \epsilon_0]$. Up to replacing $w(y, s)$ by $-w(y, s)$, we can assume that $\omega^* = 1$ in (18). Consider then $\epsilon = 2K_0 K_1 \epsilon^*$ where $K_1$ is given in Proposition 5.1 and $K_0$ will be fixed later. If

$$\epsilon^* \leq \epsilon_1 \quad \text{and} \quad \epsilon \leq \epsilon_4, \quad (243)$$
then we see that Proposition 5.1, Corollary 5.5 and (235) apply respectively with $\epsilon^*$ and $\epsilon$. In particular, there is a maximal solution $d(s) \in C^1([s^*, \infty), (-1, 1))$ such that (167) holds for all $s \in [s^*, \infty)$ where $q(y, s)$ is defined in (168) and

$$
|\theta(s^*) - \theta^*| + \|q(s^*)\|_{\mathcal{H}} \leq K_1 \epsilon^* \quad \text{with} \quad \theta^* = \frac{1}{2} \log \left( \frac{1 + d^*}{1 - d^*} \right). \quad (244)
$$

If in addition we have

$$
K_0 \geq 1 \quad \text{hence,} \quad \epsilon \geq 2K_1 \epsilon^*, \quad (245)
$$

then, we can give two definitions.

- We define first from (244) and (245) $s_1^* \in (s^*, \infty)$ such that for all $s \in [s^*, s_1^*)$,

$$
\|q(s)\|_{\mathcal{H}} < \epsilon \quad (246)
$$

and if $s_1^* < \infty$, then $\|q(s_1^*)\|_{\mathcal{H}} = \epsilon$.

- Then, we define $s_2^* \in [s^*, s_1^*)$ as the first $s \in [s^*, s_1^*)$ such that

$$
a(s) \geq \frac{b(s)}{5K_4}, \quad (247)
$$

where $K_4$ is introduced in Corollary 5.5, or $s_2^* = s_1^*$ if (247) is never satisfied on $[s^*, s_1^*)$.

We proceed in 3 steps.

- In Step 1, using (247), we integrate Eqs. (240), (241) on the time interval $[s^*, s_2^*)$ and obtain for some positive $K_6$, $\mu_6$ and $f(s)$

$$
\forall s \in [s^*, s_2^*], \quad \frac{1}{K_6} \|q\|_{\mathcal{H}}^2 \leq f \leq K_6^2 \|q\|_{\mathcal{H}}^2 \quad \text{and} \quad f' \leq -2\mu_6 f.
$$

- In Step 2, integrating Eq. (239) satisfied by $a$ on the time interval $[s_2^*, s_1^*)$, we obtain some exponential estimate.

- In Step 3, we conclude the proof by showing first that $s_1^* - s_2^* \leq \sigma_0$ for some $\sigma_0$, then $s_1^* = \infty$. Then, integrating the equation obtained in Step 1, we conclude.

In all the steps, we use the notation $C_i$ for an arbitrary constant.

**Step 1. Integration of the equations on $[s^*, s_2^*)$.** We claim the following.

**Claim 5.6.** There exist positive $\epsilon_6$, $\mu_6$, $K_6$ and $f \in C^1([s^*, s_2^*], \mathbb{R}^+)$ such that if $\epsilon \leq \epsilon_6$, then for all $s \in [s^*, s_2^*)$:

(i) $\frac{1}{2} f(s) \leq b(s) \leq 2 f(s)$ and $f'(s) \leq -2\mu_6 f(s)$,

(ii) $\|q(s)\|_{\mathcal{H}} \leq K_6 \|q(s^*)\|_{\mathcal{H}} e^{-\mu_6(s-s^*)} \leq K_1 \epsilon^* e^{-\mu_6(s-s^*)}$. 

Proof. (i) By definition of \( s^*_2 \), we see that
\[
\forall s \in [s^*, s^*_2], \quad a(s) \leq \frac{b(s)}{5K_4},
\]
(248)
where \( a(s) \) and \( b(s) \) are defined in (234). Since \([s^*, s^*_2] \subset [s^*, s^*_1]\), the interval where (246) is satisfied, we can apply Corollary 5.5. Therefore, using equations (240) and (241), we write for all \( s \in [s^*, s^*_2] \),
\[
b'(s) \leq -\frac{8}{p-1} \int_{-1}^{1} q_{x}^2 \frac{\rho}{1 - y^2} + C_1 \epsilon b(s),
\]
(249)
\[
\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho \leq -\frac{2}{5} b(s) + K_4 \int_{-1}^{1} q_{x}^2 \frac{\rho}{1 - y^2}
\]
(250)
for some \( C_1 > 0 \) and \( \epsilon^* \) small enough. We claim that
\[
f(s) = b(s) + \eta_6 \int q_1 q_2 \rho
\]
satisfies the desired property, where \( \eta_6 > 0 \) will be fixed small independent of \( \epsilon \). Using (238), we see that if \( \eta_6 \) is small enough, then we get for all \( s \in [s^*, s^*_2] \),
\[
\frac{1}{2} b(s) \leq f(s) \leq 2b(s),
\]
(251)
and using (248) and the equivalence of norms (236), we obtain for some \( C_3 > 0 \)
\[
\frac{1}{C_3} \|q(s)\|_{\mathcal{H}}^2 \leq f(s) \leq C_3 \|q(s)\|_{\mathcal{H}}^2.
\]
(252)
Then, using (249), (250) and (251), we have for all \( s \in [s^*, s^*_2] \),
\[
f'(s) \leq -\left(\frac{2}{5} \eta_6 - C_1 \epsilon \right) b(s) - \left(\frac{8}{p-1} - K_4 \eta_6 \right) \int_{-1}^{1} q_{x}^2 \frac{\rho}{1 - y^2} \leq -\frac{\eta_6}{4} b \leq -\frac{\eta_6}{8} f(s)
\]
(253)
if \( \eta_6 \) is small enough independent of \( \epsilon \), and \( \epsilon \) is small enough. Using (251), (244) and (253), this concludes the proof of (i).

(ii) Integrating Eq. (253), we get for all \( s \in [s^*, s^*_2] \), \( f(s) \leq f(s^*) e^{-\frac{\eta_6}{8}(s-s^*)} \). Using (252), this concludes the proof of Claim 5.6. \( \square \)

Step 2. Integration of the equations on \([s^*_2, s^*_1]\). We claim the following.
Claim 5.7.

(i) There exists $\varepsilon_7 > 0$ such that for all $\sigma > 0$, there exists $K_7(\sigma) > 0$ such that if $\varepsilon \leq \varepsilon_7$, then

$$
\forall s \in [s^*_2, \min(s^*_2 + \sigma, s^*_1)], \quad \|q(s)\|_{H^1} \leq K_7 \|q(s^*)\|_{H^1} e^{-\mu_6(s - s^*)} \leq K_7 K_1 \varepsilon e^{-\mu_6(s - s^*)},
$$

where $\mu_6$ has been introduced in Claim 5.6.

(ii) There exists $\varepsilon_8 > 0$ such that if $\varepsilon \leq \varepsilon_8$, then

$$
\forall s \in (s^*_2, s^*_1], \quad b(s) \leq a(s) \left(5K_4 e^{-\frac{(s-s^*_2)}{2}} + \frac{1}{4K_5}\right),
$$

where $K_4$ and $K_5$ have been introduced in Corollary 5.5.

Proof. (i) Using Eqs. (239) and (240), we see that for all $s \in [s^*_2, \min(s^*_2 + \sigma, s^*_1)]$,

$$(a + b)' \leq 3(a + b), \quad \text{hence } a(s) + b(s) \leq e^{3\sigma} \left(a(s^*_2) + b(s^*_2)\right)$$

for $\varepsilon$ small enough. Therefore, we see from (236) that $\|q(s)\|_{H^1} \leq K_4 e^{\frac{3\sigma}{2}} \|q(s^*_2)\|_{H^1}$. Using (ii) in Claim 5.6 with $s = s^*_2$ gives the conclusion.

(ii) By definition of $s^*_1$, (246) is satisfied for all $s \in [s^*_2, s^*_1]$, hence, Corollary 5.5 applies and Eqs. (239) and (240) hold.

Let us first prove that

$$
\forall s \in (s^*_2, s^*_1], \quad a(s) \geq \frac{b(s)}{5K_4},
$$

where $K_4$ is introduced in Corollary 5.5. We need to assume that $s^*_2 < s^*_1$, otherwise the set $(s^*_2, s^*_1]$ is empty. Let $g = a - \frac{b}{5K_4}$, where $a$ and $b$ are defined in (234). From Eqs. (239) and (240), we write for some $C_1 > 0$ and for all $s \in [s^*_2, s^*_1]$,

$$
a' \geq \frac{3}{2} a - C_1 \varepsilon b, \quad b' \leq C_1 \varepsilon (a + b),
$$

$$
g' = \left(a - \frac{b}{5K_4}\right)' \geq \frac{3}{2} a - C_1 \varepsilon b - \frac{C_1}{5K_4} a \leq C_1 \varepsilon (1 + 5K_4) g + a
$$

for $\varepsilon$ small enough. Since by definition of $s^*_2$, we have $g(s^*_2) \geq 0$ (remember that $s^*_2 < s^*_1$), (255) follows. Using (256) and (255), we obtain for $\varepsilon$ small enough,

$$
\forall s \in (s^*_2, s^*_1], \quad a'(s) \geq \frac{3}{2} a - 5K_4 C_1 \varepsilon a \geq a(s) \quad \text{hence } a(s) \geq e^{s - s^*_2} a(s^*_2).
$$

If $q_2(s^*_2) \equiv 0$, then $w(y, s^*_2) \equiv \kappa(d(s^*_2), y)$ by (168), and from the uniqueness of solutions to Eq. (7), we have $w(y, s) \equiv \kappa(d(s^*_2), y)$ and $q(y, s) \equiv 0$ for all $s \geq s^*_2$, hence $a(s) = b(s) = 0$ by (236) and (254) holds trivially.
Now, if \( q(s^*_2) \neq 0 \), we can define \( h = \frac{b}{a} \) for all \( s \in (s^*_2, s^*_1) \) and derive from (256) and (257) for all \( s \in (s^*_2, s^*_1) \),
\[
h' = \frac{b'a - ba'}{a^2} \leq \frac{C_1 \epsilon (a + b)a - ba}{a^2} \leq -\frac{h}{2} + C_1 \epsilon
\]
for \( \epsilon \) small enough. Integrating this equation gives
\[
b(s) \leq a(s) \left( e^{-\frac{(s - s^*_2)}{2\epsilon}} b(s^*_2) + 2C_1 \epsilon \right).
\]
Using (255) and taking \( \epsilon \) small enough gives (254) and concludes the proof of Claim 5.7. □

**Step 3. Conclusion of the proof.** We use Steps 1 and 2 to conclude the proof of Theorem 3 here. Let us first fix \( \sigma_0 > 0 \) such that
\[
5K_4 \frac{\theta_0}{2} + \frac{1}{4K_5} \leq \frac{1}{2K_5},
\]
(258)
where \( K_4 \) and \( K_5 \) are introduced in Corollary 5.5. Then, we impose the condition
\[
\epsilon = 2K_0K_1 \epsilon^*, \quad \text{where } K_0 = \max(2, K_6, K_7(\sigma_0))
\]
(259)
and the constants are defined in Proposition 5.1 and Claims 5.6 and 5.7. Finally, we fix
\[
\epsilon_0 = \min \left( 1, \epsilon_1, \frac{\epsilon_i}{2K_0K_1} \text{ for } i \in \{4, 6, 7, 8\} \right)
\]
(260)
and the constants are defined in Proposition 5.1, Corollary 5.5, Claims 5.6 and 5.7.

Now, if \( \epsilon^* \leq \epsilon_0 \), then Corollary 5.5 and Steps 1 and 2 apply. We claim that for all \( s \in [s^*, s^*_1] \),
\[
\|q(s)\|_{\mathcal{H}} \leq K_0 \|q(s^*)\|_{\mathcal{H}} e^{-\mu_6(s - s^*)} \leq K_0 \epsilon^* e^{-\mu_6(s - s^*)} = \frac{\epsilon}{2} e^{-\mu_6(s - s^*)}.
\]
(261)
Indeed, if \( s \in [s^*, \min(s^*_2 + \sigma_0, s^*_1)] \), then, this comes from (ii) of Claim 5.6 or (i) of Claim 5.7 and the definition of \( k_0(259) \).

Now, if \( s^*_2 + \sigma_0 \leq s^*_1 \) and \( s \in [s^*_2 + \sigma_0, s^*_1] \), then we have from (254) and the definition of \( \sigma_0 \),
\[
b(s) \leq \frac{a(s)}{2K_5} \text{ on the one hand. On the other hand, from (iii) in Corollary 5.5, we have } a(s) \leq K_5 b(s), \text{ hence, } a(s) = b(s) = 0 \text{ and from (236), } q(y, s) \equiv 0, \text{ hence (261) is satisfied trivially.}
\]
In particular, we have for all \( s \in [s^*, s^*_1] \), \( \|q(s)\|_{\mathcal{H}} \leq \frac{\epsilon}{2} \), hence, by definition of \( s^*_1 \), this means that \( s^*_1 = \infty \). Therefore, from (261) and (237), we have
\[
\forall s \geq s^*, \|q(s)\|_{\mathcal{H}} \leq \frac{\epsilon}{2} e^{-\mu_6(s - s^*)} \text{ and } |\theta'(s)| \leq K_4^2 e^2 \frac{\epsilon^2}{4} e^{-2\mu_6(s - s^*)}.
\]
(262)
Hence, there is \( \theta_\infty \in \mathbb{R} \) such that \( \theta(s) \to \theta_\infty \) as \( s \to \infty \) and
\[
\forall s \geq s^*, \left| \theta_\infty - \theta(s) \right| \leq C_1 \epsilon^* e^{-2\mu_6(s - s^*)} = C_2 \epsilon e^{-2\mu_6(s - s^*)}
\]
(263)
for some positive $C_1$ and $C_2$. Taking $s = s^\ast$ here and using (244), we see that $|\theta_\infty - \theta^\ast| \leq C_0 \epsilon^\ast$.

If $d_\infty = \tanh \theta_\infty$, then we see that $|d_\infty - d^\ast| \leq C_3 (1 - d^\ast 2) \epsilon^\ast$.

Using the definition of $q$ (168), (174), (262) and (263), we write

$$
\| w(s) \partial_s w(s) - \left( \kappa(d_\infty, \cdot) \right) \|_{H} \\
\leq \| w(s) \partial_s w(s) - \left( \kappa(d(s), \cdot) \right) \|_{H} + \| \kappa(d(s), \cdot) - \kappa(d_\infty, \cdot) \|_{H_0} \\
\leq \| q(s) \|_H + C |\theta_\infty - \theta(s)| \leq C_4 \epsilon^\ast e^{-\mu_6(s - s^\ast)}.
$$

This concludes the proof of Theorem 3. \(\square\)

**Appendix A. Positivity of the Lyapunov functional $E(w)$**

We prove Proposition 2.1 here. In [17], the proof is given in the “non-characteristic” case, that is when $w = w_{x_0}$ defined from some solution $u(x, t)$ to (1) where $x_0$ is a non-characteristic point of $u$. That proof naturally extends to the case where the set $[-1, 1] \times (-\log T, +\infty)$ is in the interior of the domain of definition of $w$. Let us then focus on the remaining case. Note from (16) and [17] that we only need to prove the positivity of $E(w(s))$.

Let us introduce for all $\sigma \geq 1/(T - 1/\rho)$ and $|z| < 1 + e^\sigma/\rho$,

$$
w_n(z, \sigma) = \left( 1 - \frac{e^\sigma}{\rho} \right)^{\frac{2}{p-1}} w(y, s), \quad y = \frac{z}{1 + e^\sigma/\rho} \quad \text{and} \quad s = \sigma - \log \left( 1 + \frac{e^\sigma}{\rho} \right). \tag{A.1}
$$

For a given $n$, since by definition, $w_n(y, s)$ is defined for all $|y| < 2$ for $s$ large, we see that $E(w_n(s)) \to 0$ as $s \to \infty$. Thus, since by hypothesis, we have $(w, \partial_s w)(-\log T) \in H^1 \times L^2(-1, 1)$, we obtain for all $s \in (-\log T + 2, \infty)$ and for all $n$ large enough,

$$
0 \leq E(w_n(s)) \leq E_0. \tag{A.2}
$$

One has to prove in a certain sense that $E(w_n(s_n)) \to E(w(s_0)) = E_0$ where $s_n \to s_0$.

Using [17], we have for all $s \in (-\log T + 1, \infty)$ and $n \in \mathbb{N}$,

$$
\int_1^{s+1} \int_{-1}^1 (\partial_y w_n)^2 (1 - y^2) + |w_n|^{p+1} + (\partial_s w_n)^2 + w_n^2 \rho \leq C(E_0 + 1).
$$

By convergence in energy space, we obtain for all $\delta > 0$ and $s \in (-\log T + 1, \infty)$,

$$
\int_1^{s+1} \int_{|y| < 1-\delta} (\partial_y w)^2 (1 - y^2) + |w|^{p+1} + w^2 + (\partial_s w)^2 \rho \leq C(E_0 + 1).
$$
Thus,

$$\int_{s}^{s+1} \int_{|y|<1} \left( (\partial_{y} w)^{2} (1 - y^{2}) + |w|^{p+1} + w^{2} + (\partial_{s} w)^{2} \right) \rho \leq C(E_{0} + 1). \quad (A.3)$$

We have by the Lebesgue theorem,

$$\forall s \in (- \log T + 2, \infty), \quad \int_{s}^{s+1} E(w_{n}(\tau)) \, d\tau \rightarrow \int_{s}^{s+1} E(w(\tau)) \, d\tau$$

which proves for all $s \geq - \log T + 2$, $E(w(s)) \geq 0$ from (A.2). Indeed, for all $s \in (- \log T + 2, \infty)$ and $|y| < 1$ for $n$ large (depending on $s$),

$$\left( (\partial_{y} w_{n})^{2} (1 - z^{2}) + w_{n}^{2} + |w_{n}|^{p+1} + (\partial_{s} w_{n})^{2} \right)(z, \sigma) \rho(z) \leq C_{0} \left( (\partial_{y} w)^{2} (1 - y^{2}) + w^{2} + |w|^{p+1} + (\partial_{s} w)^{2} \right)(y, s) \rho(y),$$

where $(z, \sigma)$ and $(y, s)$ are linked by (A.1), therefore, we have

$$\int_{s}^{s+1} E(w_{n}) \, d\tau_{n} \rightarrow \int_{s}^{s+1} E(w) \, d\tau.$$

Using (A.2) and the monotonicity of $E(w)$ (16), we have the conclusion.

References


